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NEXT IAS

(To be filled by candidate)

Name : ANNU KOUSIK NARASIMHA T-9179

Rol No. : OAC24MTTA003 Date of Exam.: 18-08-2024

Mobile No. : .

Exam Centre : Old Rajinder Nagar ☑ Bhopal ☐ Jaipur ☐
Mukherjee Nagar ☐ Prayagraj ☐ Online ☐

CSE (MAINS) TEST SERIES-2024

Test-09	MATHEMATICS OPTIONAL
Dated : 18-08-2024	

Time Allowed : Three Hours | Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

*There are **EIGHT** questions divided in **Two sections** and printed in **ENGLISH**.*

*Candidate has to attempt **FIVE** questions in all.*

***Question No. 1 and 5** are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.*

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	15			
2. (c)	20	20			
3. (a)	24	15			
3. (b)	27	20			
3. (c)	31	15			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	40	20			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	20			
7. (b)	68	15			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	80	20			
Grand Total					

Remarks

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Observations:

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SECTION-A

1. (a) Show that the mapping $f: V_2(R) \to V_3(R)$ defined as $T(a,b) = (a+b, a-b, b)$ is a linear transformation from $V_2(R)$ into $V_3(R)$. Find the range, rank, null space and nullity of T. (10)

Given : $T(a,b) = (a+b, a-b, b)$

(i) Range;

$T : V_2 \to V_3$; Basis of $V_2 = \{(1,0), (0,1)\}$

$T(1,0) = (1,1,0)$; $T(0,1) = (1,-1,1)$

Range = span $\{(1,1,0), (1,-1,1)\}$

$= p(1,1,0) + q(1,-1,1)$

$= (p+q, p-q, q)$

(ii) Rank = dimension (Range)

Range = span $\{(1,1,0), (1,-1,1)\}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \text{Echelon form}$$

∴ Rank = No of non zero rows

$= 2$

∴ Rank $(T) = 2$

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(iii) Nullspace: $\{(a,b) \in V_2 : T(a,b) = (0,0,0)\}$

$\Rightarrow a+b=0, \; a-b=0, \; b=0$

$\Rightarrow a=b=0$

$\therefore$ Nullspace $(T) = \{(0,0)\}$

(iv) As per Rank - Nullity Theorem,

$\text{Rank}(T) + \text{Nullity}(T) = \dim(V_2(\mathbb{R}))$

$\Rightarrow 2 = 2 + \text{Nullity}(T)$ $\quad (\because \dim(V_2)=2, \; \text{Rank}=2)$

$\therefore$ Nullity $(T) = 0$

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1. (b) If A and B are matrices of the same order say n, show that the relation $AB - BA = I_n$ doesnot hold good. (10)



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1. (c) Evaluate: $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$ (10)

Let $L = \lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$

$\Rightarrow L = \lim_{x\to 1}\frac{(x-1)}{\left(\cot\frac{\pi x}{2}\right)}$. This is in

$\frac{0}{0}$ (Indeterminate form);

Using L-Hospital rule

$\left(\lim\frac{f}{g} = \lim\frac{f'}{g'}\right)$

$\Rightarrow \lim_{x\to 1}\frac{x-1}{\cot\frac{\pi x}{2}} = \lim_{x\to 1}\frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}\left(\cot\frac{\pi x}{2}\right)}$

$= \lim_{x\to 1}\frac{1}{\left(\frac{\pi}{2}\right)\times -\operatorname{cosec}^2\left(\frac{\pi x}{2}\right)}$

$= \frac{-2}{\pi}$

$\therefore \lim_{x\to 1}(x-1)\tan\frac{\pi x}{2} = -\frac{2}{\pi}$

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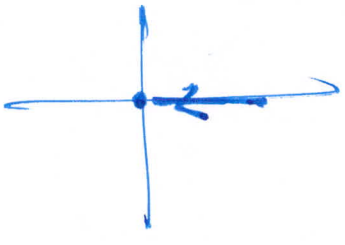
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1. (d) Show that the function $f(x,y)=\begin{cases}1 & ,xy\neq 0\\ 0 & ,xy=0\end{cases}$ is not continuous at (0,0). (10)

A function $f(x,y)$ is continuous at (a,b) if $\forall\ \varepsilon>0$, there exist some $\delta>0$ such that $|f(x,y)-f(a,b)|<\varepsilon$ for all points in the disc with radius δ.

(i) Approaching along $y=0$ (x-axis)

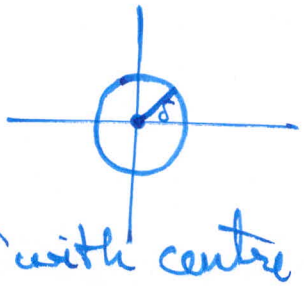


Then $f(x,y)=0$ $\forall\ x$.

$(\because y=0)$

$\therefore$ f is continuous on $y=0$.

(ii) Approaching along some path obliquely to $(0,0)$:



Let $\varepsilon=1/2$.

Consider a disc of radius δ with centre $(0,0)$.

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Then at the origin, $f(0,0) = 0$,

At all other points on the disc, $f(x,y) = 1$ (other than axes)

$(\because x, y \neq 0)$

$\therefore |f(x,y) - f(0,0)| = 1 > \varepsilon$

$\therefore$ There exist no $\delta > 0$ such that

$|f(x,y) - f(0,0)| < \varepsilon$

Hence the fn f is not continuous at $(0,0)$

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1. (e) Show that the locus of the centres of spheres of coaxial system is a straight line. (10)

Let $S = x^2+y^2+z^2+2ux+2vy+2wz+d_1$

and $P = 2ax+2by+2cz+d_2$

Then the ~~locus of~~ spheres $S+\lambda P$ is form of coaxial spheres, λ is parameter.

$$S+\lambda_1 P = x^2+y^2+z^2+x(2u+2a\lambda_1)+y(2v+2b\lambda)$$
$$+z(2w+2c\lambda_1)+(d_1+\lambda d_2)=0 \quad \hookrightarrow (1)$$

is the form of coaxial system.

Now, the centre of (1) is

$$(-u-a\lambda_1, -v-b\lambda_1, -w-c\lambda_1) = (x_1, y_1, z_1) \text{ (say)}$$

$$\Rightarrow \lambda_1 = \frac{x_1+u}{-a}, \quad \lambda_1 = \frac{y_1+v}{-b}, \quad \lambda_1 = \frac{z_1+w}{-c}$$

The centre of (1) lies on

$$\boxed{\frac{x_1+u}{a} = \frac{y_1+v}{b} = \frac{z_1+w}{c}} \quad \text{---} (2)$$

Hence for any parameter λ, the centre of spheres line on straight line (2).

Hence Proved.

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2. (a) Let V be complex vector space of 3×3 skew symmetric matrices with complex entries i.e. $V=\{A \epsilon M_{3\times 3}(\mathbb{C}) | A^t=-A\}$. Let $B=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$. Define a linear transformation $T:V\to V$ by $T(A)=BA-AB$. Find the eigen values and eigenvectors of T. (15)



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2. (b) Prove that the minimum radius vector of the curve $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$ is of length $a + b$.

(15)



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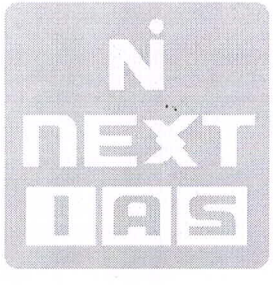
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2. (c) If the plane $lx+my+nz=p$ passes through the extremities of three conjugate semidiameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, prove that $a^2l^2+b^2m^2+c^2n^2=3p^2$. (20)



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3. (a) Find the inverse of A by Gauss-Jordan method where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. (15)

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$[A|I] \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 3 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 + 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 7 & -6 & 2 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 3 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 + 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 3 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & -1 & 0 & +3 & -3 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 = -R_3]{R_2 = -R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] = [I|A^{-1}]$$

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$\therefore A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ is the required inverse of A.

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3. (b) Trace the curve $y = \dfrac{2x-3}{x^2-3x+2}$. (20)

(i) Checking for symmetricity

→ Not symmetric about y axis

$\because y(x) \neq y(-x)$

→ Not symmetric about x-axis

(∵ y is of odd degree)

→ Not symmetric about origin

(ii) $y = \dfrac{2x-3}{x^2-3x+2} = \dfrac{2x-3}{(x-1)(x-2)}$

As $x \to 1^- \Rightarrow y \to -\infty$

As $x \to 1^+ \Rightarrow y \to +\infty$

As $x \to 2^- \Rightarrow y \to -\infty$

As $x \to 2^+ \Rightarrow y \to +\infty$

As $x \to +\infty$, $y \to 0$

As $x \to -\infty$, $y \to 0$

For $x < 1$, $y < 0$

At $x = 3/2$, $y = 0$; At $x = 0$, $y = -3/2$

(iii) Asymptotes : $y = 0, x = 1, x = 2$

(By putting highest powers of x, y to zero in $y(x-1)(x-2) = 2x-3$)

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(iv) Finding the slope at various pts/regions.

$$\frac{dy}{dx} = \frac{(x^2-3x+2)(2)-(2x-3)(2x-3)}{(x^2-3x+2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(2x^2-6x+4)-(4x^2+9-12x)}{(x^2-3x+2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x^2+6x-5}{(x^2-3x+2)^2}$$

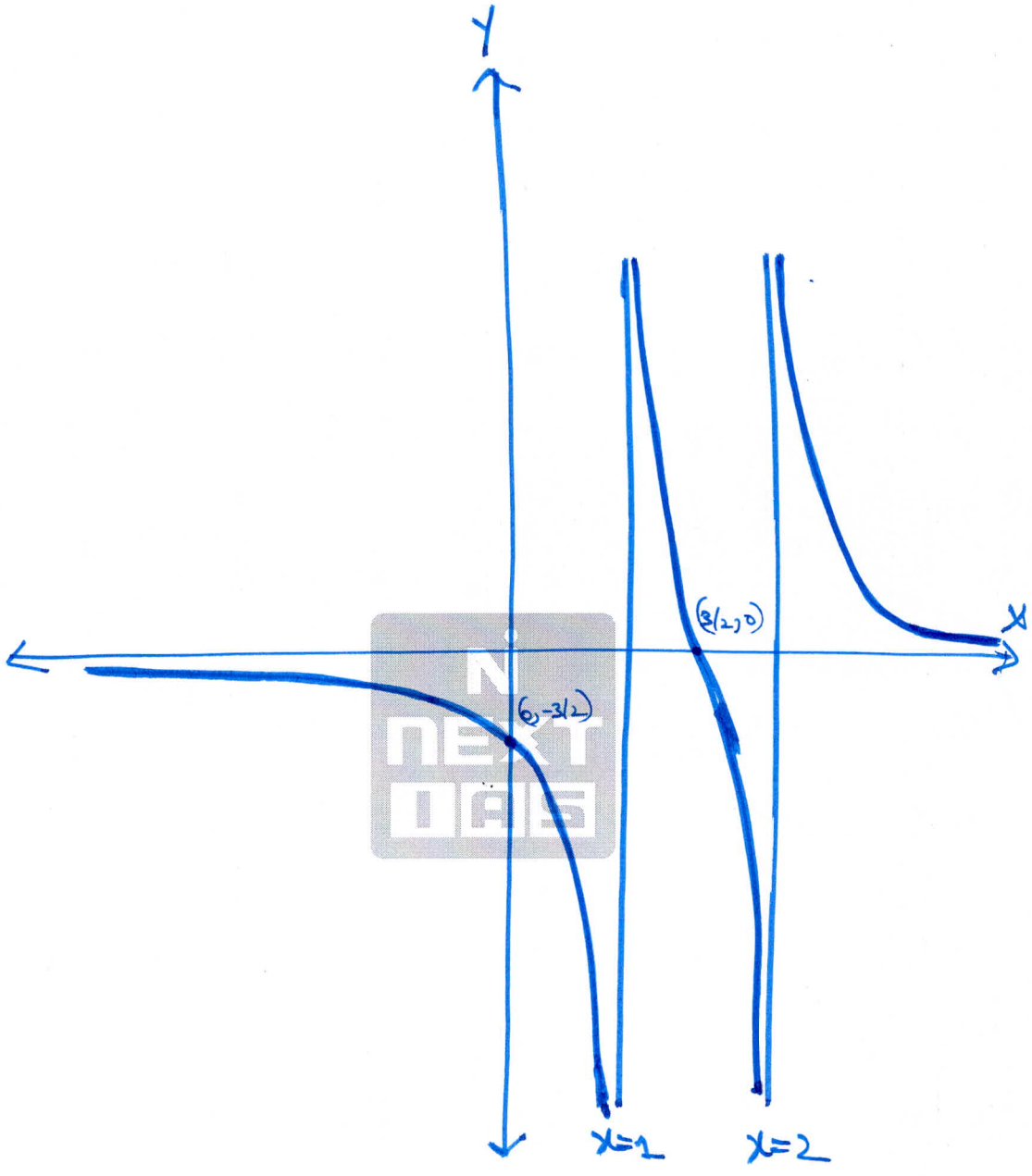
For numerator, $\Delta < 0$ & $a < 0$

$\therefore$ The numerator keeps same sign.

$\therefore \frac{dy}{dx} < 0 \quad \forall x$

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Is the reqd eqn of curve $y = \frac{3x-1}{x^2-3x+2}$

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3. (c) Find the surface generated by a straight line which intersects the lines $x+y=0=z, x-y-z=0=x+y-2a$ and the parabola $y=0=x^2-2az$. (15)

Given lines: $L_1 \equiv x+y=0=z$ and

$L_2 \equiv x-y-z=0=x+y-2a$

Any plane through L_1 : $(x+y)+\lambda z=0$

Any plane through L_2 : $(x-y-z)+\mu(x+y-2a)=0$

The intersection of 2 planes is a line,

the straight line intersects the parabola

$y=0=x^2-2az$

Substituting $y=0$ in P_1, we get

$x+\lambda z=0$ ——— (1).

Substituting $y=0$ in P_2, we get

$(x+z)+\mu(x-2a)=0$

$(\mu+1)x+z=2a\mu$ ——— (2).

Finding the value of x & z from (1) & (2),

$x=-\lambda z$

$\Rightarrow (\mu+1)(-\lambda z)+z=2a\mu$.

$\Rightarrow (-\lambda\mu-\lambda)z+1=2a\mu$

$\Rightarrow 2a\mu-1=z(-\lambda\mu-\lambda) \Rightarrow z=\dfrac{1-2a\mu}{\lambda(\mu+1)}$

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Also $x = -\lambda z$

$\Rightarrow x = (-\lambda)\left(\frac{1-2a\mu}{\lambda(\mu+1)}\right) = \frac{2a\mu-1}{\mu+1}$

$\Rightarrow \boxed{x = \frac{2a\mu-1}{\mu+1}}$; $\boxed{z = \frac{1-2a\mu}{\lambda(\mu+1)}}$

$x + \lambda z = 0$ — (1)

$(x-z) + \mu(x-2a) = 0$ — (2)

$(\mu+1)x - z = 2a\mu$

$\Rightarrow (\mu+1)(-\lambda z) - z = 2a\mu$

$\Rightarrow -\lambda\mu z - \lambda z - z = 2a\mu$

$\Rightarrow z(-\lambda\mu - \lambda - 1) = 2a\mu$

$\Rightarrow z = \frac{-2a\mu}{\lambda\mu+\lambda+1}$, $x = -\lambda z$

$\Rightarrow x = \frac{2a\lambda\mu}{\lambda\mu+\lambda+1}$. Now $x^2 - 2az = 0$

$\Rightarrow \frac{4a^2\lambda^2\mu^2}{(\lambda\mu+\lambda+1)^2} - \frac{2a(-2a\mu)}{\lambda\mu+\lambda+1} = 0$

$\Rightarrow \frac{4a^2\lambda^2\mu^2}{\lambda\mu+\lambda+1} + 4a^2\mu = 0$

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$$\frac{\lambda^2\mu}{\lambda\mu+\lambda+1} + 1 = 0 \Rightarrow \lambda^2\mu+\lambda\mu+\lambda+1=0.$$

$$\lambda\mu(\lambda+1)+(\lambda+1)=0$$

$$\Rightarrow (\lambda\mu+1)(\lambda+1)=0$$

$$\Rightarrow \lambda\mu+1=0 \quad (\because \lambda \neq -1 \text{ as } \lambda \text{ is parameter})$$

$$\Rightarrow \left(\frac{-x+y}{z}\right)\left(\frac{-x+y+z}{x+y-2a}\right)+1=0$$

$$\Rightarrow \frac{(x+y)(x-y-z)}{z(x+y-2a)} + 1 = 0$$

$$\Rightarrow x^2-xy-xz+xy-y^2-yz+xz+yz-2az=0.$$

$$\Rightarrow x^2-y^2-2az=0$$

$$\text{or} \quad \boxed{x^2-y^2=2az} \text{ is the reqd eqn of surface generated by line.}$$

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4. (a) Let $\beta=\{(1,1,0),(1,0,1),(0,1,1)\}$ and $\beta'=\{(2,1,1),(1,2,1),(-1,1,1)\}$ be the two ordered bases of R^3. Then find a matrix representing the linear transformation $T:R^3\to R^3$ which transforms β into β'. Use this matrix representation to find $T(x)$, where $x=(2,3,1)$. (15)

To find : $T : T(\beta) = \beta'$

$\Rightarrow T(1,1,0) = (2,1,1)$; — (1)

$T(1,0,1) = (1,2,1)$; — (2)

$T(0,1,1) = (-1,1,1)$; — (3).

We know that $T(p,q,r) = pT(1,0,0) + qT(0,1,0) + rT(0,0,1)$

Performing (2) − (3) :

$\Rightarrow T(1,-1,0) = (2,1,0)$ — (4)

(4) + (1) $\Rightarrow T(2,0,0) = (4,2,1)$

$\Rightarrow \boxed{T(1,0,0) = (2,1,1/2)}$ → (I)

Performing (2) − (1) :

$\Rightarrow T(0,-1,1) = (-1,1,0)$ — (5)

Performing (5) + (3)

$\Rightarrow T(0,0,2) = (-2,2,1)$

$\Rightarrow \boxed{T(0,0,1) = (-1,1,1/2)}$ → (II)

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~~By using (1) & (3): T(1,0,1) = (3,0,0) — (6)~~

~~(6) + (2) ⇒ T(2,0,0) = (4,2,1)~~

~~⇒~~ From (5): T(0,-1,1) = (-1,1,0)

T(0,1,-1) = (-1,1,1)

⇒ T(0,+2,0) = (0,0,1)

⇒ **T(0,1,0) = (0,0,1/2)** — (III)

~~T(1,0,0) = (2,1,1)~~ **T(a,b,c) = (2a-c, a+c, a/2 + b/2 + c/2)** is the reqd L-T

From (I), (II) & (III),

$$\text{Matrix}(T) = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 1 \\ 1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$T(2,3,1) = (\text{Matrix}(T))\,(2\ 3\ 1)^T$$

$$T(2,3,1) = (3,3,3)$$

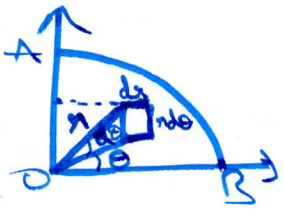
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4. (b) The arc of the cardiod $r = a(1+\cos\theta)$ included between $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$ is rotated about the line $\theta = \frac{\pi}{2}$. Find the volume of the solid of revolution. (15)



Volume of revolution
= 2 × (Volume of revolution of the segment ADB for angle π)

Volume of differential element is

$$dV = (r\,dr\,d\theta) \times 2\pi \times r\sin\theta$$

$$dV = 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$V = \int dV = \int_0^{\frac{\pi}{2}} \int_0^{a(1+\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$\Rightarrow V = 2\pi \times \int_0^{\frac{\pi}{2}} \left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)} \sin\theta\, d\theta$$

$$\Rightarrow V = \frac{2\pi a^3}{3} \int_0^{\frac{\pi}{2}} \sin\theta\,(1+\cos\theta)^3\, d\theta$$

$$\Rightarrow V = \frac{2\pi a^3}{3} \int_0^{\frac{\pi}{2}} \sin\theta\,(1+\cos^3\theta+3\cos\theta+3\cos^2\theta)\, d\theta$$

$$\Rightarrow V = \frac{2\pi a^3}{3} \int_0^{\frac{\pi}{2}} \sin\theta + \sin\theta\cos^3\theta + 3\cos\theta\sin\theta + 3\cos^2\theta\sin\theta\, d\theta.$$

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$$\Rightarrow V = \frac{2\pi a^3}{3} \times \frac{15}{4} \qquad \left(\because \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}\right)$$

$$\Rightarrow V = \frac{2\pi a^3}{3} \times \frac{15}{4} = \frac{5\pi a^3}{2}$$

Total volume $= 2 \times V = 5\pi a^3$

$\therefore$ Volume $= 5\pi a^3$ cubic units.

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4. (c) Find the locus of points from which three mutually perpendicular lines can be drawn to intersect the conic $z = 0,\ ax^2 + by^2 = 1$. (20)

Let the reqd pt be (x_1, y_1, z_1)

The line through (x_1, y_1, z_1) is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \text{---(1)}$$

$z = 0$ lies on (1)

$$\Rightarrow \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{-z_1}{n}$$

$$\Rightarrow x = x_1 + \frac{lz_1}{n},\ y = y_1 + \frac{mz_1}{n}$$

$$ax^2 + by^2 = 1$$

$$\Rightarrow a\left(x_1 - \frac{lz_1}{n}\right)^2 + b\left(y_1 - \frac{mz_1}{n}\right)^2 = 1$$

$$\Rightarrow a(nx_1 - lz_1)^2 + b(ny_1 - mz_1)^2 = n^2$$

$$\Rightarrow a(n^2x_1^2 + l^2z_1^2 - 2lnx_1z_1) + b(n^2y_1^2 + m^2z_1^2 - 2mny_1z_1) = n^2.$$

$$\Rightarrow (an^2x_1^2 + al^2z_1^2 - 2alnx_1z_1) + (bn^2y_1^2 + bm^2z_1^2 - 2bmny_1z_1) = n^2$$

$$\Rightarrow n^2(ax_1^2 + by_1^2 - 1) + l^2(az_1^2) + m^2(bz_1^2)$$

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$-2a lmx_1z_1 - 2b mn y_1z_1 = 0.$

Coeff of l^2 + Coeff of m^2 + Coeff of $n^2 = 0$.

$\Rightarrow ax_1^2 + by_1^2 - 1 + az_1^2 + bz_1^2 = 0.$

$\Rightarrow ax_1^2 + by_1^2 + z_1^2(a+b) = 1$

$\Rightarrow$ The locus of the points are

$$\boxed{ax^2 + by^2 + (a+b)z^2 = 1}$$

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SECTION-B

5. (a) Solve the differential equation $y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0$. Verify that the obtained solution satisfies the given differential equation. (10)

The equation is of the form $y(M')dx + x(N')dy = 0$

Choosing the Integrating factor as $\frac{1}{Mx-Ny}$

$Mx-Ny = \left(x^2y^2+2x^3y^3\right)-\left(x^2y^2-x^3y^3\right) = 3x^3y^3$

$IF = \frac{1}{Mx-Ny} = \frac{1}{3x^3y^3}$ — (1)

Multiplying the eqn with (1) on both sides,

$$\frac{xy^2+2x^2y^3}{3x^3y^3}dx + \frac{x^2y-x^3y^2}{3x^3y^3}dy = 0.$$

$$\Rightarrow \left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy = 0.$$

$= M_1dx + N_1dy = 0$. Now

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} = -\frac{1}{3x^2y^2}$$

The solution is given by $u+v=c$ where

$v = \int M_1dx$, $u = \int N_1 - \frac{\partial v}{\partial y}dy$.

$$\int M_1dx = \int \frac{x^{-2}}{3y}+\frac{2}{3x}dx = \frac{1}{3y}\left(\frac{x^{-1}}{-1}\right)+\frac{2}{3}\ln x$$

$$\boxed{v = -\frac{1}{3xy}+\frac{2}{3}\ln x}$$

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$\frac{\partial v}{\partial y} = \frac{1}{3xy^2}$

$N_1 - \frac{\partial v}{\partial y} = \frac{1}{3xy^2} - \frac{1}{3y} = \frac{1}{3xy^2} - \frac{xy}{3xy^2} = \frac{-1}{3y}$

$\int N_1 - \frac{\partial v}{\partial y}\, dy = \int \frac{-1}{3y}\, dy = -\frac{1}{3}\ln y = u$

∴ The solution is

$\frac{2}{3}\ln x - \frac{1}{3}\ln y - \frac{1}{3xy} = C$

(or) $\frac{2}{3}\ln x - \ln y - \frac{1}{xy} = C$

$\Rightarrow \boxed{\ln\left(\frac{x^2}{y}\right) - \frac{1}{xy} = C}$ ——— (I).

Differentiating (I),

$\frac{1}{\left(\frac{x^2}{y}\right)} \times \frac{y(2x\,dx) - x^2 dy}{y^2} + \frac{1}{x^2y^2}(y\,dx + x\,dy) = 0.$

$\Rightarrow \frac{2xy\,dx - x^2 dy}{x^2 y} + \frac{y\,dx + x\,dy}{x^2y^2} = 0$

$\Rightarrow 2xy^2 dx - x^2 dy + y\,dx + x\,dy = 0$. Multiplying by xy on both sides,

$2x^2y^3 dx + x^2y^2 dx + x^2 y\,dy - x^3y^2 dy = 0.$

$\Rightarrow y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Is the required eqn. Hence proved.

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5. (b) In what direction from the point (1, 1, −1) is the directional derivative of $f = x^2 - 2y^2 + 4z^2$ is a maximum? Also find the value of this maximum directional derivative. (10)

The directional derivative is maximum in the direction $\frac{\nabla f}{|\nabla f|}$ at the pt (1,1,1)

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\Rightarrow \nabla f = 2xi - 4yj + 8zk$$

$$\frac{[\nabla f]}{|\nabla f|} = \frac{2xi - 4yj + 8zk}{\sqrt{4x^2 + 16y^2 + 64z^2}} = \frac{xi - 2yj + 4zk}{\sqrt{x^2 + 4y^2 + 16z^2}}$$

At (1,1,−1), $\frac{\nabla f}{|\nabla f|} = \frac{i - 2j - 4k}{\sqrt{1 + 4 + 16}} = \frac{i - 2j - 4k}{\sqrt{21}}$

is the direction in which the directional derivative is maximum.

Value of directional derivative $= \nabla f \cdot \frac{\nabla f}{|\nabla f|}$

$$\Rightarrow \text{directional derivative} = (2i - 4j - 8k) \cdot \frac{(i - 2j - 4k)}{\sqrt{21}}$$

$$\Rightarrow \text{dir. derivative} = \frac{2 + 8 + 32}{\sqrt{21}} = \frac{42}{\sqrt{21}}$$

$$= 2\sqrt{21}$$

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$\therefore$ directional derivative is maximum in the direction $\frac{i-2j-4k}{\sqrt{21}}$ from $(1,1,-1)$

Value of directional derivative $= 2\sqrt{21}$

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5. (c) Show that the curve $\vec{x}(t) = t\hat{i} + \left(\frac{1+t}{t}\right)\hat{j} + \left(\frac{1-t^2}{t}\right)\hat{k}$ lies in a plane. (10)

The curve lies in plane ⇒ any 3 vector $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are coplanar.

Let $\bar{x}_1 = t_1 i + \left(1+\frac{1}{t_1}\right)j + \left(-t_1 + \frac{1}{t_1}\right)k$

$\bar{x}_2 = t_2 i + \left(1+\frac{1}{t_2}\right)j + \left(-t_2 + \frac{1}{t_2}\right)k$

$\bar{x}_3 = t_3 i + \left(1+\frac{1}{t_3}\right)j + \left(-t_3 + \frac{1}{t_3}\right)k$

Calculating $[\bar{x}_1\ \bar{x}_2\ \bar{x}_3]$

$$\begin{bmatrix} t_1 & 1+\frac{1}{t_1} & -t_1+\frac{1}{t_1} \\ t_2 & 1+\frac{1}{t_2} & -t_2+\frac{1}{t_2} \\ t_3 & 1+\frac{1}{t_3} & -t_3+\frac{1}{t_3} \end{bmatrix}$$

$$= t_1\left[\left(1+\frac{1}{t_2}\right)\left(-t_3+\frac{1}{t_3}\right) + \left(1+\frac{1}{t_1}\right)\left[-t_2t_3 + \frac{t_3}{t_2} + t_2t_3 - \frac{t_2}{t_3}\right] - \left(1+\frac{1}{t_3}\right)\left(t_2+\frac{1}{t_2}\right)\right]$$

$$+ \left(\frac{1}{t_1} - t_1\right)\left[t_2 + \frac{t_2}{t_3} - t_3 - \frac{t_3}{t_2}\right]$$

$$= t_1\left(-t_3 + \frac{1}{t_3} - \frac{t_3}{t_2} + \frac{1}{t_2t_3}\right) + \left(1+\frac{1}{t_1}\right)\left(\frac{t_3}{t_2} - \frac{t_2}{t_3}\right) + t_2 + \frac{t_2}{t_3} - \frac{1}{t_2} - \frac{1}{t_2t_3}$$

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$$+\left(\frac{1}{t_1}-t_1\right)\left(t_2-t_3+\frac{t_2}{t_3}-\frac{t_3}{t_2}\right)$$

$$=\left(-t_3t_1+\frac{t_1}{t_3}-\frac{t_1t_3}{t_2}+\frac{t_1}{t_2t_3}+t_1t_2+\frac{t_1t_2}{t_3}-\frac{t_1}{t_2}-\frac{t_1}{t_2t_3}\right)$$

$$+\left(\frac{t_3}{t_2}-\frac{t_2}{t_3}+\frac{t_3}{t_1t_2}-\frac{t_2}{t_1t_3}\right)$$

$$+\left(\frac{t_2}{t_1}-\frac{t_3}{t_1}+\frac{t_2}{t_1t_3}-\frac{t_3}{t_1t_2}-t_1t_2+t_1t_3-\frac{t_1t_2}{t_3}+\frac{t_1t_3}{t_2}\right)$$

$$=0.$$

$\therefore [\bar{x}_1\ \bar{x}_2\ \bar{x}_3]=0$

for any 3 vectors

$\therefore$ The curve $x(t)$ lies in a plane.

Hence proved.

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5. (d) A chain of n equal uniform rods is smoothly jointed together and suspended from its one end A_1. A horizontal force $\vec{P}$ is applied to the other end A_{n+1} of the chain. Find the inclinations of the rod to the downward vertical line in the equilibrium configuration. (10)

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5. (e) The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$, where λ and μ are constants; find the polar equation of the path of the particle and also it's radial and transverse acceleration in terms of r and θ only. (10)

Given that $v_r = \lambda r^2$ and $v_\theta = \mu\theta^2$

$$v_r = \frac{dr}{dt} = \lambda r^2 \; ; \; v_\theta = r\frac{d\theta}{dt} = \mu\theta^2$$

$$\Rightarrow \frac{dr}{dt} = \lambda r^2 \; , \; \frac{d\theta}{dt} = \frac{\mu\theta^2}{r}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{dr/dt}{d\theta/dt} = \frac{\lambda r^2}{\mu\theta^2/r} = \frac{\lambda r^3}{\mu\theta^2}$$

$$\Rightarrow \mu r^{-3}\,dr = \lambda\theta^{-2}\,d\theta$$

$$\Rightarrow \frac{dr}{\lambda r^3} = \frac{d\theta}{\mu\theta^2}$$. Integrating on both sides,

$$\frac{1}{\lambda}\int r^{-3}\,dr - \frac{1}{\mu}\int\theta^{-2}\,d\theta = \int 0$$

$$\Rightarrow \frac{1}{\lambda}\left(\frac{r^{-2}}{-2}\right) - \frac{1}{\mu}\left(\frac{\theta^{-1}}{-1}\right) = c$$

$$\Rightarrow \frac{-1}{2\lambda r^2} + \frac{1}{\mu\theta} = c$$

$$\Rightarrow \boxed{\frac{1}{\mu\theta} - \frac{1}{2\lambda r^2} = c}$$ is the reqd eqn of polar curve.

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radial acceleration:

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$\frac{dr}{dt} = \lambda r^2 \Rightarrow \frac{d^2r}{dt^2} = 2\lambda r \frac{dr}{dt} = 2\lambda r(\lambda r^2)$$

$$= 2\lambda^2 r^3$$

$$\therefore \frac{d^2r}{dt^2} = 2\lambda^2 r^3$$

$$a_r = 2\lambda^2 r^3 - r\left(\frac{\mu^2\theta^4}{r^2}\right)$$

$$\Rightarrow \boxed{a_r = 2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}}$$ is the reqd radial acceleration.

Transverse acceleration

$$a_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right), \quad r^2\frac{d\theta}{dt} = \mu r\theta^2$$

$$\frac{d}{dt}\left(\mu r\theta^2\right) = \mu\left(\frac{dr}{dt}\theta^2 + 2r\theta\frac{d\theta}{dt}\right)$$

$$= \mu\left(\theta^2 \times \lambda r^2 + 2r\theta \times \frac{\mu\theta^2}{r}\right) = \mu\left(\lambda r^2\theta^2 + 2\mu\theta^3\right)$$

$$a_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \mu\left(\lambda r\theta^2 + \frac{2\mu\theta^3}{r}\right)$$

$$\therefore \boxed{a_\theta = \mu\left(\lambda r\theta^2 + \frac{2\mu\theta^3}{r}\right)}$$ is the reqd transverse acceleration.

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6. (a) Reduce the differential equation $xp^2 - 2yp + x + 2y = 0$, to clairaut's form and obtain its complete primitive. Also, determine a singular solution of the given differential equation. (20)

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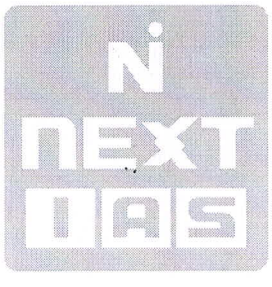
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6. (b) Show that $\text{grad}(A \cdot B) = (B \cdot \nabla)A + (A \cdot \nabla)B + B \times \text{curl } A + A \times \text{curl } B$. (15)

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6. (c) A heavy uniform rod of length $2a$, rests with its end in contact with two smooth inclined planes, of inclination α and β to the horizon. If θ be the inclination of the rod to the horizon, prove by principle of virtual work, that

$$\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta).$$ (15)

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7. (a) Solve $x^2y'' + xy' - y = 0$, given that $x + \frac{1}{x}$ is one integral by using the method of reduction of order. (20)

$x^2y'' + xy' - y = 0 \Rightarrow \boxed{y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0}$

Comparing with $y'' + Py' + Qy = R$,

we see $P = \frac{1}{x}$, $Q = \frac{-1}{x^2}$, $R = 0$.

Given that one solution: $u = x + \frac{1}{x}$.

The other soln is given by the DE:

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{1}{x} + \frac{2}{\left(x+\frac{1}{x}\right)} \times \left(1 - \frac{1}{x^2}\right)\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{1}{x} + \frac{2x}{x^2+1} \times \frac{x^2-1}{x^2}\right)\frac{dv}{dx} = 0.$$

Let $\frac{dv}{dx} = k \Rightarrow \frac{d^2v}{dx^2} = \frac{dk}{dx}$

$$\Rightarrow \frac{dk}{dx} + \left(\frac{1}{x} + \frac{2(x^2-1)}{x(x^2+1)}\right)k = 0.$$

$$\Rightarrow \frac{dk}{k} + \left(\frac{1}{x} + \frac{2(x^2-1)}{x(x^2+1)}\right)dx = 0.$$

$$\Rightarrow \frac{dk}{k} + \left(\frac{x^2+1+2x^2-2}{x(x^2+1)}\right)dx = 0.$$

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$\Rightarrow \frac{dk}{k} + \left(\frac{3x^2-1}{x(x^2+1)}\right)dx = 0.$

Integrating, $\int \frac{3x^2-1}{x(x^2+1)}\,dx.$

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

$\Rightarrow \int \frac{3\tan^2\theta - 1}{\tan\theta \times \sec^2\theta} \times \sec^2\theta\,d\theta = \int 3\tan\theta\cot\theta\,d\theta$

$= 3\ln|\sec\theta| - \ln|\sin\theta| = \ln|\sec^3\theta| - \ln|\sin\theta|$

$= \ln\left|\frac{\sec^3\theta}{\sin\theta}\right| = \ln\left|\frac{\sec^4\theta}{\frac{\sin\theta}{\cos\theta}}\right| = \ln\left|\frac{(1+x^2)^2}{x}\right|$

$\Rightarrow \ln k + \ln\left|\frac{(1+x^2)^2}{x}\right| = c_1$

$\Rightarrow \frac{k(1+x^2)^2}{x} = c_2$

$\Rightarrow k = \frac{c_2 x}{(1+x^2)^2} = \frac{dv}{dx}$

$\Rightarrow dv = \frac{c_2 x\,dx}{(1+x^2)^2}$. Integrating on both sides,

$v = \int dv = \frac{c_2}{2}\int \frac{2x\,dx}{(1+x^2)^2} = c_3\int \frac{2x\,dx}{(1+x^2)^2}$

Let $x^2 = t \Rightarrow 2x\,dx = dt.$

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$\therefore v = C_3 \int (dt)(t^{-2}) = C_3 \times \frac{t^{-1}}{-1}$

$= \frac{-C_3}{(1+x^2)} = \frac{C_4}{(1+x^2)}$

$\therefore v = \frac{C_4}{1+x^2}$

The final solution is given by

$y = uv$

$\Rightarrow y = \left(x + \frac{1}{x}\right)\left(\frac{C_4}{1+x^2}\right) = \frac{C_4}{x}$

$\therefore y = \frac{k}{x}$ or $xy = k$ is the reqd solution of the DE.

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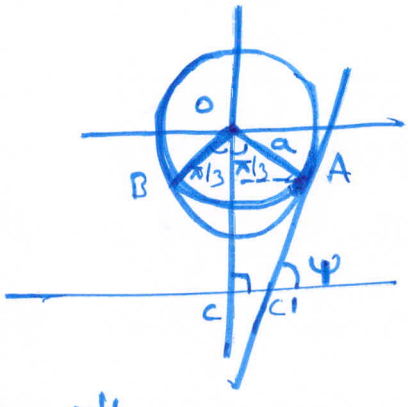
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7. (b) Find the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two third of the circumference of the pulley. (15)

$\angle AOC = \frac{\pi}{3} = \angle BOC$

Contact angle $= \frac{2}{3} \times 2\pi$

$= \frac{4\pi}{3}$.



Let the angle at end A with directrix be ψ.

$\angle OAC' = \pi/2 \Rightarrow \angle AC'C = 2\pi - \left(\frac{\pi}{3} + \frac{\pi}{2} + \frac{\pi}{2}\right)$

$= \frac{2\pi}{3}$

$\Rightarrow \psi = \pi - \angle AC'C = \frac{\pi}{3}$.

$\therefore \psi = \frac{\pi}{3}$.

The value $s_A = a \sin\frac{\pi}{3} = \frac{\sqrt{3}a}{2}$.

$s_A = c \ln|\sec\psi + \tan\psi|$

$s_A = c \ln|\sqrt{3} + 2| = \frac{\sqrt{3}a}{2}$

$\Rightarrow c = \frac{\sqrt{3}\,a}{2\ln(2+\sqrt{3})}$

The length of string upto A $= S_A = c \tan\psi$

$\Rightarrow S_A = c \times \tan\pi/3 = \sqrt{3}\,c$

$\therefore S_A = \frac{3a}{2\ln(2+\sqrt{3})}$

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Total length of string

$$L = \left(\frac{4\pi}{3}\right)a + 2S_A$$

$$\Rightarrow L = \frac{4\pi a}{3} + \frac{3a}{\ln(2+\sqrt{3})}$$

$$\Rightarrow L = a\left(\frac{4\pi}{3} + \frac{3}{\ln(2+\sqrt{3})}\right)$$

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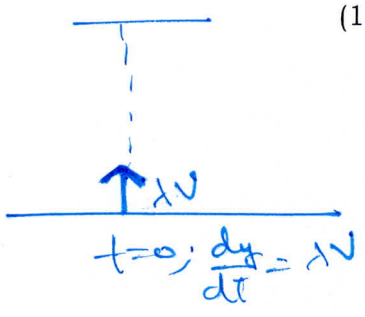
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7. (c) A particle of mass m projected vertically under gravity, resistance of air being mk times velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}\left[\lambda - \log(1+\lambda)\right]$ where V is the terminal velocity of the particle and λV is the initial vertical velocity. (15)



$$m\frac{d^2y}{dt^2} = -mg - mk\frac{dy}{dt}$$

(Force balancing)

$$\Rightarrow \frac{d^2y}{dt^2} = -g - k\frac{dy}{dt}.$$

$$v = \frac{dy}{dt}$$

Multiplying both sides with $2\frac{dy}{dt}$,

$$2\frac{dy}{dt}\frac{d^2y}{dt^2} = -2g\frac{dy}{dt} - 2k\left(\frac{dy}{dt}\right)^2$$

$$\frac{dv}{dt} = -g - kv \Rightarrow \frac{dv}{(g+kv)} + dt = 0.$$

Integrating, $\frac{\ln(g+kv)}{k} + t = c_1$

at $t=0$, $\vec{v} = \lambda V$

$$\Rightarrow c_1 = \ln(g + \lambda kV)$$

$$\therefore \ln(g+kv) + kt = k\ln(g+\lambda kv)$$

$$\Rightarrow -\ln(g+kv) + k\ln(g+\lambda kv) = kt.$$

$$\Rightarrow \ln\left(\frac{(g+\lambda kv)^k}{g+kv}\right) = kt$$

$$\Rightarrow (g+\lambda kv)^k = (g+kv)e^{kt}$$

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$$\Rightarrow \left(g + k\frac{dy}{dt}\right)^k = \left(g + k\frac{dy}{dt}\right)e^{kt}$$

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8. (a) Solve the following differential equation:

$$(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1. \qquad (15)$$



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8. (b) Evaluate $\iint_S \left(ax^2 + by^2 + cz^2\right) dS$ over the sphere $x^2 + y^2 + z^2 = 1$ using the divergence theorem. (15)



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8. (c) Two particles of masses m_1 and m_2 moving in coplanar parabolas round the sun, collide at right angles and coalesce when their common distance from the sun is R. Show that the subsequent path of the combined particle is an ellipse of major axis $\dfrac{(m_1+m_2)^2 R}{2m_1m_2}$. (20)

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SPACE FOR ROUGH WORK

$S+\lambda P \quad \frac{1}{3x^2y}dx+\frac{1}{3xy^2}dy = \frac{-1}{3xy}\left(\frac{1}{x}dx+\frac{1}{y}dy\right)$

$x^2+y^2+z^2+2ux+2vy+2wz+\lambda(ax+by+cz+d)$

$\ln xy$

$xy^2+2x^2y^3 \; ; \; x^2y-x^3y^2$

$-3-3(-2)\left(x^2y^2+2x^3y^3-\left(x^2y^2-x^3y^3\right)\right.$

$0-3(1)$

$1-3(0) \qquad 3x^3y^3$

$36-4(-2)(-5) \qquad \frac{xy^2(1+2xy)}{3x^3y^3}$

$=-4<0$

$\frac{1}{3x^2y^2}$; $\frac{1}{2}$

$(t_2-\frac{1}{t_2})\left(x^{1+\frac{1}{2xy}}\right)2=rw2$

$(t+\frac{1}{t_3})$



$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}2&0&-1\\1&0&1\\1/2&1/2&1/2\end{bmatrix}\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$

$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}2&0&-1\\1&0&1\\1/2&1/2&1/2\end{bmatrix}$

$=\begin{bmatrix}2&1&-1\end{bmatrix}$

$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$

$=\begin{bmatrix}2&1&-1\end{bmatrix}$