

NEXT IAS

(To be filled by candidate)

Name of Candidate : Kamishak AggarwalRoll No. : MT 24 MATL 1011 Date of ExaminationMobile Number : 1 mail 31/7Exam Centre : Old Rajinder Nagar ☐Online ☒**CSE (MAINS) TEST SERIES-2024**Test 008**MATHEMATICS
OPTIONAL**

Dated : 14-07-2024

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are EIGHT questions divided in Two sections and printed in ENGLISH.

Candidate has to attempt FIVE questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question cum Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	15			
2. (c)	20	20			
3. (a)	24	15			
3. (b)	27	15			
3. (c)	30	20			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	40	20			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	15			
6. (b)	57	20			
6. (c)	61	15			
7. (a)	64	20			
7. (b)	68	15			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	80	20			
Grand Total					

Remarks

Observations:

1. The first observation is that the data is not normally distributed. This is evident from the skewness and kurtosis values. The skewness is positive, indicating a right-skewed distribution, and the kurtosis is high, indicating a heavy-tailed distribution.

2. The second observation is that the data is not stationary. This is evident from the time series plot, which shows a clear upward trend over time.

3. The third observation is that the data is not independent. This is evident from the autocorrelation function (ACF) plot, which shows significant autocorrelation at lag 1 and lag 2.

4. The fourth observation is that the data is not white noise. This is evident from the Ljung-Box test, which shows a significant p-value, indicating that the data is not white noise.

5. The fifth observation is that the data is not Gaussian. This is evident from the Q-Q plot, which shows a clear deviation from the normal distribution line.

SECTION-A

(10)

1. (a) Find all the units of $\mathbb{Z}\sqrt{-5}$.

$$\mathbb{Z}\sqrt{-5} = a + b\sqrt{-5}, \quad a, b \in \mathbb{Z}$$

let $a + b\sqrt{-5}$ be a unit.

then also unity is 1

$$(a + b\sqrt{-5})(c + d\sqrt{-5}) = 1$$

$$(ac - 5bd) + \sqrt{-5}(ad + bc) = 1$$

$$\Rightarrow ac = 5bd + 1$$

$$ad = -bc$$

$$\Rightarrow \frac{c}{d} = \frac{5d}{-c}$$

$$c^2 = -5d^2$$

$$\text{or } \frac{d}{c} = \pm \frac{1}{\sqrt{-5}}$$

$$\frac{a}{b} = \frac{5d}{c} = \pm \frac{5}{\sqrt{-5}} = \pm \sqrt{-5}$$

$$a = \pm b\sqrt{-5}$$

$$\begin{aligned} a + b\sqrt{-5} &= b\sqrt{-5}(\pm 1 + 1) \\ &= 0, 2b\sqrt{-5} \end{aligned}$$

$$\cancel{ac = 5b\left(-\frac{bc}{a}\right) + 1 \Rightarrow a^2c + 5b^2c = a}$$

$$\cancel{(a^2 + 5b^2)c = a}$$

$$(a + b\sqrt{-5})(c + d\sqrt{-5}) = 1$$

$$(a - b\sqrt{-5})(c - d\sqrt{-5}) = 1$$

$$(ac - 5bd) + \sqrt{-5}(-ad - bc) = 1$$

$$(a^2 + 5b^2)(c^2 + 5d^2) = 1$$

$$\Rightarrow a^2 + 5b^2 = 1 = c^2 + 5d^2 \quad (\because a, b, c, d \in \mathbb{Z})$$

$$\Rightarrow a^2 = 1; 5b^2 = 0$$

$$\Rightarrow a = \pm 1$$

Hence ± 1 are the only units of $\mathbb{Z}\sqrt{-5}$

Q. 24

What is the value of $\sin^{-1}(\sin \frac{\pi}{6})$?



Show that every convergent sequence is bounded but the converse may not be true. (10)

Part 1. let $\{a_n\}$ be a convergent sequence.

$$\text{i.e. } |a_n - l| < \epsilon \quad \forall n \geq m$$

$$-\epsilon + l < a_n < \epsilon + l \quad \forall n \geq m$$

$$\{a_n\} = a_1, a_2, \dots, a_{n-1}, a_n, \dots$$

$$\text{let } \alpha = \max \{a_1, a_2, \dots, a_{n-1}\}$$

$$\text{also } |a_n| < l + \epsilon \quad \forall n \geq m$$

$$\text{let } \beta = \max \left\{ |\alpha|, |l + \epsilon|, |l - \epsilon| \right\}$$

$$\Rightarrow |a_n| < |\beta| \quad \forall n \in \mathbb{N}$$

Converse is not true.

$$\text{eg } a_n = (-1)^n$$

$$|a_n| < 2$$

$\Rightarrow a_n$ is bounded

$$\text{but } \{a_n\} = -1, 1, -1, \dots$$

oscillates i.e. not convergent

1. (d)

Prove that the converse of Lagrange's theorem is false.

(10)

Lagrange's theorem

$$\text{Let } G \text{ is a group and } H < G$$

$$\text{then } o(H) \mid o(G)$$

Converse.

$$\exists H \text{ such that } o(H) \mid o(G)$$

i.e. all divisors of $o(G)$

order subgroup is possible.

Proof

$\rightarrow G$ is cyclic

$$\text{Let } G = \langle a \rangle$$

$$o(G) = m = o(a)$$

$$\text{Let } n \mid m.$$

$$\text{Consider } H = \langle a^n \rangle ; \text{ let } o(H) = p = o(a^n)$$

$\Rightarrow$

$$a^{np} = e$$

$$\Rightarrow o(a) \leq np$$

$$\Rightarrow m \leq np \quad \text{--- i}$$

Also

$$o(a) = m$$

$$\Rightarrow a^m = e$$

$$\Rightarrow (a^n)^{\frac{m}{n}} = e$$

$$\Rightarrow o(a^n) \leq m/n$$

$$\Rightarrow p \leq m/n \Rightarrow m \geq np \quad \text{--- ii}$$

from i, ii
 $m = np$.

i.e. $O(H) = m/n$ or n/m

• n is any general divisor of m

so every divisor of $O(G)$ will
make a subgroup.

Hence converse true in finite
cyclic group.

1. [9]

Prove that a basic feasible solution to a linear programming problem must correspond to an extreme point of the set of all feasible solutions. (10)

or



2. (a)

If G is a group s.t. $\frac{G}{Z(G)}$ is cyclic, where $Z(G)$ is centre of G then show that G is abelian. (15)

$$\frac{G}{Z(G)} = \{zg \mid g \in G\}$$

$\frac{G}{Z(G)}$ is cyclic

$$\text{let } \frac{G}{Z(G)} = \langle za \rangle \quad a \in G$$

$$\text{let } a, b, c \in G$$

$$\text{To prove } bc = cb.$$

$$\text{Now } b, c \in G$$

$$\Rightarrow zb, zc \in \frac{G}{Z(G)}$$

$$\text{i.e. } zb = (za)^{n_1} = (za)(za) \dots (za) = za^{n_1}$$

$$zc = (za)^{n_2} \quad (\because Z \text{ is normal in } G)$$

$$= za^{n_2}$$

$$(zb)(zc) = z(bc)$$

$$(zb)(zc) = (za^{n_1})(za^{n_2}) = za^{n_1}a^{n_2}$$

$$= za^{n_1+n_2}$$

$$= za^{n_2+n_1}$$

$$= za^{n_2}a^{n_1}$$

$$= (za^{n_2})(za^{n_1})$$

$$= (zc)(zb)$$

$$= z(cb)$$



$$x(bc) = z(cb)$$

$$(bc)(cb)^{-1} \in Z$$

$$(bc)(cb)^{-1}(cb) = (cb)(bc)(cb)^{-1}$$

$$(bc) = (cb)(bc)(cb)^{-1}$$

$$(bc)(cb)^{-1}(bc)^{-1} = (bc)^{-1}(bc)(cb)^{-1}$$

$$(bc)(b^{-1}c^{-1})(c^{-1}b^{-1}) = (cb)^{-1}$$

$$(bc)(cb)(bc)^{-1} = (cb)$$

$$(bc)(cb) = (cb)(bc)$$

$$bc^2b = cb^2c$$

$$(bc)(cb)^{-1}b = b(bc)(cb)^{-1}$$

$$(bc)(b^{-1}c^{-1})b = cb^{-1}$$

$$(bc)(b^{-1}c^{-1})c = c(bc)(b^{-1}c^{-1})$$

$$bc = c(bc)(c)$$

$$b = cbc$$

$$(bc)(b^{-1}) = c(bc)(b^{-1}c^{-1})$$

$$(bc)c$$

$$b \in Z_b \quad (a \in M_a) ; \quad c \in Z_c$$

$$b = z_1 a^{n_1} ; \quad c = z_2 a^{n_2}$$

$$\begin{aligned}
 bc &= z_1 a^{n_1} z_2 a^{n_2} \\
 &= z_1 z_2 a^{n_1} a^{n_2} \\
 &= z_1 z_2 a^{n_1 + n_2} = z_1 z_2 a^{n_2 + n_1} \\
 &= z_1 z_2 a^{n_2} a^{n_1} \\
 &= z_1 z_2 c b \\
 &= z_2 z_1 c b \\
 &= z_2 c z_1 b \\
 &= cb
 \end{aligned}$$

$$\Rightarrow bc = cb.$$

(b) Let a, b be real numbers and $0 < \lambda < 1$. Put $x_1 = a$, $x_2 = b$ and $x_{n+2} = \lambda x_n + (1-\lambda)x_{n+1}$. Show that the sequence $\{x_n\}$ converges and find its limit. (15)

$$x_{n+2} = \lambda x_n + (1-\lambda)x_{n+1}$$

~~$$x_{n+1} = \lambda x_{n-1} + (1-\lambda)x_n$$~~

$$x_{n+2} - x_{n+1} = \lambda x_n + (1-\lambda)x_{n+1} - x_{n+1}$$

$$x_{n+2} - x_{n+1} = \lambda(x_n - x_{n+1})$$

$$x_{n+1} - x_n = \lambda(x_{n-1} - x_n)$$

$$\vdots$$

$$\left. \begin{aligned} x_4 - x_3 &= \lambda(x_2 - x_3) \\ x_3 - x_2 &= \lambda(x_1 - x_2) \end{aligned} \right\}$$

Multiplying

$$(x_{n+2} - x_{n+1}) \cancel{(x_3 - x_2)} = \lambda^{n-1} (-1)^{n-1} (x_1 - x_2)$$

$$x_{n+2} - x_{n+1} = (-1)^{n-1} (\lambda^{n-1}) (a-b)$$

$$x_{n+2} - x_{n+1} = (-1)^n \lambda^{n-1} (b-a)$$

Non monotonic.

$$x_3 = \lambda x_1 + (1-\lambda)x_2$$

$$x_3 = \lambda a + (1-\lambda)b$$

$$x_3 - b = \lambda(a-b)$$

$$= b + \lambda(a-b)$$

$$x_4 = \lambda x_2 + (1-\lambda)x_3$$

$$= \lambda b + (1-\lambda)[b + \lambda(a-b)]$$

$$x_4 = b + \lambda(a-b) - \lambda^2(a-b)$$

$$x_4 - x_3 = \lambda(x_2 - x_3) = \lambda^2(b-a)$$

$$x_4 = x_3 + \text{ive}$$

Adding

$$(x_{n+2} - x_2) = \lambda(x_1 - x_{n+1})$$

$$x_{n+2} - x_{n+1} = \lambda a + b - x_{n+1}$$

2. (c)

Show that an isolated singular point z_0 of a function $f(z)$ is a pole of order m iff $f(z)$ can be written in the form $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and

non-zero at z_0 . Moreover, $\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 1$. (20)

$$\text{let } f(z) = \frac{\phi(z)}{(z-z_0)^m}; \phi(z) \text{ is analytic}$$

To prove $f(z)$ has pole at z_0 of order m

$$\text{Now } \phi(z) = (z-z_0)^m f(z)$$

$\phi(z)$ is analytic

$\therefore$ Taylor expansion around z_0

$$\phi(z) = \phi(z_0) + (z-z_0)\phi'(z_0) + \frac{(z-z_0)^2}{2!}\phi''(z_0) + \dots$$

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} = \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi'(z_0)}{(z-z_0)^{m-1}} + \dots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!(z-z_0)} + \frac{\phi^{(m)}(z_0)}{m!} + \frac{\phi^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \dots$$

Now principal part has m terms in $f(z)$
 has z_0 is pole of order m

Residue is coeff of $\frac{1}{z-z_0}$

$$= \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

Now f has pole of order m

To prove $f = \frac{\phi(z)}{(z-z_0)^m}$, ϕ analytic

$$f = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \dots + \frac{b_1}{z-z_0}$$

$$+ a_0 + a_1(z-z_0) + \dots$$

$$f(z-z_0)^m = b_m + b_{m-1}(z-z_0) + \dots + b_1(z-z_0)^{m-1}$$

$$+ a_0(z-z_0)^m + a_1(z-z_0)^{m+1} + \dots$$

RHS is power series

which converges absolutely

$\therefore$ RHS is analytic function
i.e. let $\phi(z)$

$$\therefore f(z)(z-z_0)^m = \phi(z)$$

$$\Rightarrow f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

4. (a)

Prove that corresponding to every divisor of the order of a cyclic group there is a unique subgroup of that order. (10)

let n be order of cyclic group G

let m be divisor of n

$$G = \langle a \rangle ; a^n = e$$

$$H_1 = \langle b \rangle$$

$$o(H_1) = o(b) = o(c)$$

$$\textcircled{*} H_2 = \langle c \rangle$$

$$= o(H_2) = m$$

To prove $H_1 = H_2$.

$$H_1 \leq G ; H_2 \leq G$$

$$\textcircled{*} H_1 = \langle a^{r_1}, a^{r_2}, \dots, a^{r_{k_1}} \rangle = \langle a^{p_1} \rangle$$

$$H_2 = \langle a^{s_1}, a^{s_2}, \dots, a^{s_{k_2}} \rangle = \langle a^{p_2} \rangle$$

$\therefore n$ is finite $\therefore m$ is finite.

now without loss of generality
let r_1, s_1 is smallest amongst
all r_i, s_i respectively.

~~let~~

Now

$$r_1 = s_1 q + R$$

$$p_1 = kp_2 + e \quad e < p_2$$

$$mp_1 = mkp_2 + me$$

$$a^{mp_1} = a^{mkp_2 + me} = (a^{mkp_2}) \cdot a^{me}$$

$$(a^{p_1})^m = ((a^{p_2})^m)^k a^{me}$$

$$e = e \cdot a^{me} \Rightarrow a^{me} = e$$

$$(\because o(a^{p_1}) = o(a^{p_2}) = m) \quad me < mp_2$$

$$\text{but } o(a) = mp_2$$

$$\Rightarrow \frac{me}{o} = 0 \quad me = 0$$

$$e = 0$$

$$\text{i.e. } p_1 = kp_2$$

$$H_1 = \langle a^{p_1} \rangle = \langle a^{kp_2} \rangle \subseteq H_2$$

$$\text{but } o(H_1) = o(H_2)$$

$$\Rightarrow H_1 = H_2$$

(b)

A necessary and sufficient condition for the integrability of a bounded function f is that to every $\epsilon > 0$, there corresponds a partition P such that the corresponding oscillatory sum $w(P, f) < \epsilon$. (15)

let function f is integrable

$$U(P, f) - L(P, f) < \epsilon.$$

$\rightarrow$ Let P_1, P_2 be two partitions

$$U(P_1, f) \leq \int_a^b f dx + \epsilon/2 \quad (\text{property of infimum})$$

$$L(P_2, f) > \int_a^b f dx - \epsilon/2 \quad (\text{property of supremum})$$

let $P = P_1 \cup P_2$

$$U(P, f) < U(P_1, f) < \int_a^b f dx + \epsilon/2$$

$$L(P, f) > L(P_2, f) > \int_a^b f dx - \epsilon/2$$

$$U(P, f) - L(P, f) < \int_a^b f dx - \int_a^b f dx + \epsilon$$

$$< \epsilon \quad \left(\because \int_a^b f dx = \int_a^b f dx \right)$$

$$\Rightarrow w(P, f) < \epsilon.$$

$$\text{let } U(P, f) - L(P, f) < \epsilon$$

To prove $f \in R[a, b]$

$$L(P, f) < \int_a^b f dx < \int_a^{\bar{b}} f dx < U(P, f)$$

$$\text{We have } \int_a^b f dx - \int_a^{\bar{b}} f dx < U(P, f) - L(P, f) < \epsilon \text{ (given)}$$

$$\Rightarrow \int_a^b f dx - \int_a^{\bar{b}} f dx < \epsilon$$

ϵ can be arbitrarily close to 0

$$\Rightarrow \int_a^b f dx = \int_a^{\bar{b}} f dx$$

$$\Rightarrow f \in R[a, b]$$

$$x_1 + \frac{x_2}{2} + \frac{x_3}{2} \leq 1,$$

4. (c)

Use penalty method to minimise $Z = x_1 + 2x_2 + x_3$, subject to

$$\frac{3}{2}x_1 + 2x_2 + x_3 \geq 8,$$

$$x_1, x_2, x_3 \geq 0$$

(20)

$$\text{Min } Z = x_1 + 2x_2 + x_3 + 0s_1 + 0e_1 + Ma_1$$

$$\text{Max } Z = -x_1 - 2x_2 - x_3 + 0s_1 + 0e_1 - Ma_1$$

s.t.

$$x_1 + \frac{x_2}{2} + \frac{x_3}{2} + s_1 = 1$$

$$\frac{3}{2}x_1 + 2x_2 + x_3 - e_1 + a_1 = 8$$

$$x_1, x_2, x_3, s_1, e_1, a_1 \geq 0$$

IBFS

$$s_1 = 1; a_1 = 8$$

$$x_1 = x_2 = x_3 = e_1 = 0$$

x_B	C_B	x_1	x_2	x_3	s_1	e_1	a_1	b	θ
0	s_1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	1	2
-M	a_1	$\frac{3}{2}$	2	1	0	-1	1	8	4
Z_j		$-\frac{3M}{2}$	$-2M$	$-M$	0	M	$-M$	$-8M$	
C_j		$1 + \frac{3M}{2}$	$2 + 2M$	$-1 + M$	0	$-M$	0		
			$\uparrow$						
2	x_2	2	1	1	2	0	0	2	
-M	a_1	$-\frac{5}{2}$	0	-1	-4	-1	1	4	
Z_j		$4 + \frac{5M}{2}$	2	$-2 + M$	$1 + 4M$	M	$-M$	$4 - 4M$	
C_j		$3 - \frac{5M}{2}$	0	$1 - M$	$1 - 4M$	$-M$	0		

$$x_2 \leq 0$$

⇒ optimality reached

$$x_2 = 2, \quad x_1 = x_3 = 0$$

but x_1 remain in basis

hence no feasible and ~~opt~~ solution exist.



$$\text{All } c_j \leq 0$$

$\Rightarrow$ optimality reached

$$x_2 = 2 \quad ; \quad x_1 = x_3 = 0$$

but a_1 remain in basis

hence no feasible ~~and no opt~~
solution exist.



5. (a)

By eliminating the arbitrary functions f and g from $z = f(x^2 - y) + g(x^2 + y)$, form partial differential equation.

$$\frac{\partial z}{\partial x} = p = 2x[f'(x^2 - y) + g'(x^2 + y)]$$

$$\frac{\partial z}{\partial y} = q = f'(x^2 - y) + g'(x^2 + y)$$

$$\frac{\partial^2 z}{\partial y^2} = r = f''(x^2 - y) + g''(x^2 + y)$$

$$t = \frac{1}{2x} p$$

$$p = 2xt$$

5. (b)

If the function $f(x) = \sin x$ is approximated by a polynomial of degree 9 that interpolates f at 10 points in the interval $[0, 1]$, how large is the error on this interval?

$$\text{Error} = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i)$$

here $n = 9$;

$$\text{Error} \leq \frac{|f^{(9)}(\xi)|}{10!} \prod_{i=0}^9 (x - x_i)$$

$$\leq \frac{|\sin \xi|}{10!} (1)$$

$$\text{Error} \leq \frac{1}{10!}$$

as $(x - x_0)(x - x_1) \dots$
maximum
can be
 $(x_9 - x_0)(x_9 - x_1) \dots$
 $(x_9 - x_0)$
 $= (1)(1) \dots (1)$

5. (c)

The velocity of a train which starts from rest is given in the following table.
 The time is in minutes and velocity is in km/hours.

T(min.)	2	4	6	8	10	12	14	16	18	20
V(km/hr)	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0

Estimate approximately the total distance (km) run in 20 minutes by using composite Simpson's 1/3rd rule. (10)

T	0	2	4	6	8	10	12	14	16	18	20
V	0	16	28.8	40	46.4	51.2	32	17.6	8	3.2	0
n	0	1	2	3	4	5	6	7	8	9	10

$$\text{Area} = \int v dt$$

$$= \frac{h}{3} [v_0 + v_n + 2(v_{\text{even}}) + 4(v_{\text{odd}})]$$

$$= \frac{2}{3} [0 + 0 + 2(28.8 + 46.4 + 32 + 8)$$

$$+ 4(16 + 40 + 51.2 + 17.6 + 3.2)]$$

$$= 494.933$$

$$= 494.9 \times \frac{\text{km}}{\text{hr}} \times \text{min}$$

$$494.9 \times \frac{\text{km}}{\text{hr}} \times \frac{1}{60} \text{ hr}$$

$$\text{Distance} = 8.2 \text{ km}$$

5. (d)

Find the stream function of the two-dimensional motion due to two equal sources and an equal sink situated midway between them.

$$\begin{array}{ccc} \text{•} & \text{•} & \text{•} \\ m & -m & m \\ (-1,0) & (0,0) & (1,0) \end{array}$$

$$w = m \log z - m \log(z-1) - m \log(z+1)$$

$$w = \phi + i\psi$$

$$\psi = m \left(\tan^{-1} \frac{y}{x} - \tan^{-1} \frac{y}{x-1} - \tan^{-1} \frac{y}{x+1} \right)$$

$$= m \left(\tan^{-1} \frac{y}{x} - \tan^{-1} \left(\frac{\frac{y}{x-1} + \frac{y}{x+1}}{1 - \frac{y}{x-1} \cdot \frac{y}{x+1}} \right) \right)$$

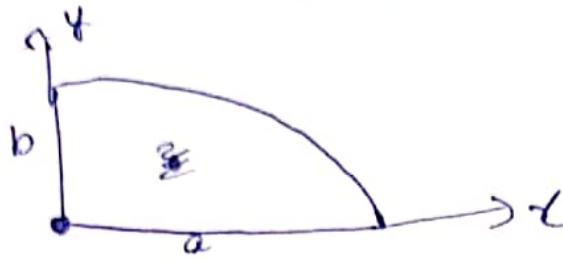
$$= m \left(\tan^{-1} \frac{y}{x} - \tan^{-1} \left(\frac{2yx}{x^2 - y^2 - 1} \right) \right)$$

$$= m \left(\tan^{-1} \left(\frac{\frac{y}{x} - \frac{2yx}{x^2 - y^2 - 1}}{1 + \frac{y}{x} \cdot \frac{2yx}{x^2 - y^2 - 1}} \right) \right)$$

$$\psi = m \left[\tan^{-1} \left(\frac{-yx^2 - y^3 - y}{x^3 + x^2} \right) \right]$$

5. (c)

Show that at the centre of a quadrant of an ellipse, the principal axes in its plane are inclined at an angle $\frac{1}{2} \tan^{-1} \left(\frac{4ab}{\pi a^2 - b^2} \right)$. (10)



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\tan 2\theta = \frac{2F}{B-A}$$

$$A = I_x = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sigma dx dy y^2$$

$$\frac{\sigma b}{3} \int_0^a b^3 \left(1 - \frac{x^2}{a^2}\right)^{3/2} dx$$

$$x = a \sin \theta$$

$$\frac{\sigma}{3} \int_0^{\pi/2} b^3 \cos^3 \theta a \cos \theta d\theta$$

$$\frac{\sigma b^3 a}{3} \int_0^{\pi/2} \cos^4 \theta d\theta$$

$$= \frac{\sigma b^3 a}{16} \pi$$

$$= \sigma \left(\frac{\pi a b^3}{4} \right) \frac{b^2}{4}$$

$$A = \frac{M b^2}{4}$$

Similarly $B = \frac{Ma^2}{4}$

$$F = \sigma \iint (dx dy) xy$$

$$x=0 \text{ to } a, y=0 \text{ to } b$$

$$= \int_0^a \int_0^b \frac{\sigma b^2}{2} \left(1 - \frac{x^2}{a^2}\right) dx dy$$

$$= \frac{b^2}{2} \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{b^2}{2} \left[x - \frac{x^3}{3a^2} \right]_0^a$$

$$= \frac{b^2}{2} \left(\frac{a^2}{2} - \frac{a^3}{3a^2} \right) = \frac{\sigma a^2 b^2}{8} = \frac{Mab}{2\pi}$$

$$\tan 2\theta = \frac{2 \left(\frac{Mab}{2\pi} \right)}{M \left(\frac{a^2 - b^2}{4} \right)}$$

$$= \frac{4ab}{\pi(a^2 - b^2)}$$

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{4ab}{\pi(a^2 - b^2)} \right)$$

8. (a)

Obtain the Boolean Function $F(x, y, z)$ based on the table given below. Then simplify $F(x, y, z)$ and draw the corresponding Gate network.

x	y	z	$f(x, y, z)$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

(15)

using min term

$$f(x, y, z) = \bar{x}yz + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}y\bar{z}$$

~~$x\bar{y}z + x\bar{y}\bar{z} + \bar{x}y\bar{z}$~~

$$f(x, y, z) = xyz + xy\bar{z} + x\bar{y}z + \bar{x}y\bar{z}$$

$$= xy(z + \bar{z}) + x\bar{y}z + \bar{x}y\bar{z}$$

$$= xy + x\bar{y}z + \bar{x}y\bar{z}$$

$$x(y + \bar{y}z) + \bar{x}y\bar{z}$$

$$x(y + \bar{y}z) + \bar{x}y\bar{z}$$

$$x(y + \bar{y})(y + \bar{z}) + \bar{x}y\bar{z}$$

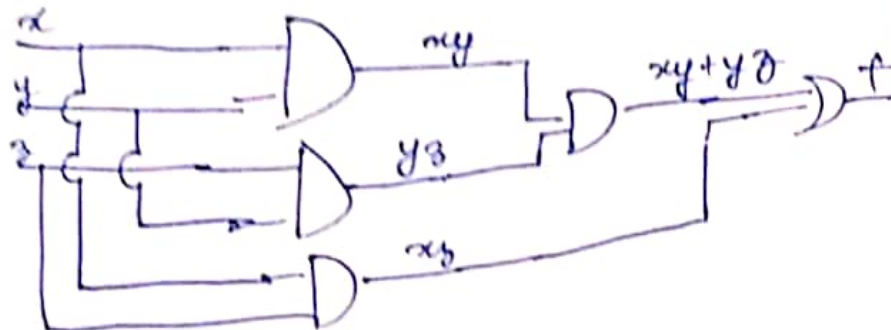
$$x(y + \bar{z}) + \bar{x}y\bar{z}$$

$$xy + xz + \bar{x}y\bar{z}$$

$$xy + xz + \bar{x}y\bar{z}$$

$$xy + xz + \bar{x}y\bar{z} = xy + xz + \bar{x}y\bar{z}$$

$$F(x, y, z) = xy + yz + zx$$



Prove that the necessary and sufficient conditions that the vortex lines may be at right angles to the stream lines are $u, v, w = \mu \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$ where μ and ϕ are function of x, y, z, t .

(15)

$$(i) \quad \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$(ii) \quad \frac{d\xi_1}{\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}} = - \frac{d\xi_2}{\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}} = \frac{d\xi_3}{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}$$

if ii at right angle

$$\Rightarrow u \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + v \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + w \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = 0$$

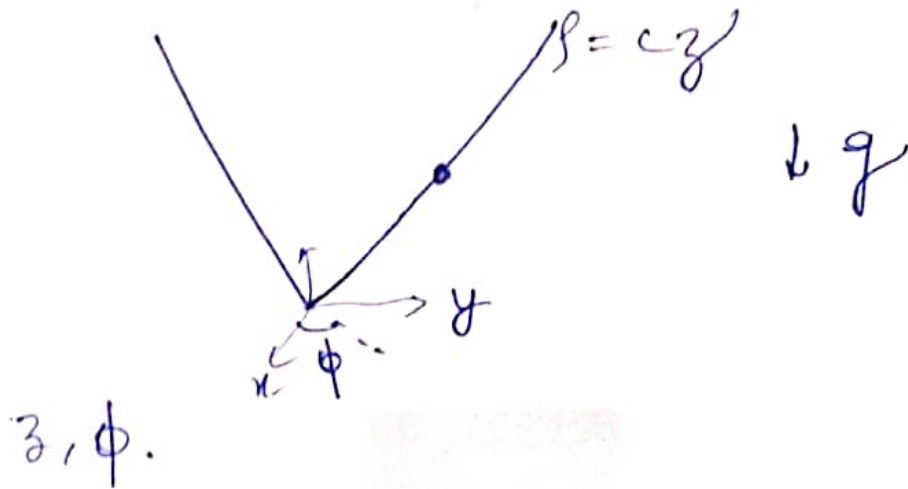
which is the necessary and sufficient condition for $u dx + v dy + w dz$ to be a perfect differential

$$u dx + v dy + w dz = \mu d\phi$$

$$\Rightarrow u = \mu \frac{\partial \phi}{\partial x}; v = \mu \frac{\partial \phi}{\partial y}; w = \mu \frac{\partial \phi}{\partial z}$$

8. (c)

Consider a mass m which is constrained to move on the frictionless surface of a vertical cone $\rho = cz$ in a uniform gravitation field g vertically down. Set up Hamilton's equations z and ϕ using as generalized coordinates. Show that for any given solution there are maximum and minimum heights z_{max} and z_{min} between which the motion is confined. (20)



ϕ is angle with positive z axis

$$\rho = cz = c r \cos \theta$$

$$\Rightarrow \cos \theta = 1/c$$

θ is angle with positive z axis.

$$V = -mgz$$

$$KE = \frac{1}{2} m v^2$$

$$\vec{r} = r \cos \theta (\hat{z}) + r \sin \theta (\hat{\phi}) + z(\hat{r})$$

$$\text{now } v^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2$$

$$v^2 = \dot{r}^2 + \dot{r}^2 \sin^2 \theta + \dot{z}^2/c^2$$

$$KE = \frac{1}{2} m \left(\dot{r}^2 + \dot{r}^2 \sin^2 \theta + \dot{z}^2/c^2 \right)$$

$$H = KE + V$$

$$= \frac{1}{2} m (\dot{z}^2 + \dot{\theta}^2 + \dot{\phi}^2) - mgz$$

$$H = \frac{1}{2} m (\dot{z}^2 + \dot{\theta}^2 + \dot{\phi}^2) - mgz$$

it ~~doesn't~~ contain

$$L = KE - V$$

$$L = \frac{1}{2} m (\dot{z}^2 + \dot{\theta}^2 + \dot{\phi}^2) + mgz$$

It doesn't explicitly contain t

$\therefore H$ is possible

Also

$$\frac{\partial L}{\partial \dot{z}} = p_z = m \dot{z} \Rightarrow \dot{z} = \frac{p_z}{m}$$

$$\frac{\partial L}{\partial \dot{\theta}} = p_\theta = m \dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{m}$$

$$H = \frac{1}{2} m \left(\frac{p_z^2}{m^2} + \frac{p_\theta^2}{m^2} \right) - mgz$$

$$\frac{\partial H}{\partial z} = -p_z = \frac{1}{m} \left[\frac{-p_\theta^2 (2)}{m^2 z^3} \right] - H_1$$

$$\frac{\partial H}{\partial \theta} = -p_\theta = 0 - H_2$$

$$\frac{\partial H}{\partial p_z} = \dot{z} = \frac{p_z}{m} - H_3$$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{p_\theta}{m} - H_4$$

$$\dot{z} = \frac{p_z}{m}$$

$$\ddot{z} = \frac{\dot{p}_z}{m} = \frac{p_0^2}{2mc^2 z^3}$$

Also $\dot{p}_0 = 0 \Rightarrow p_0 = \alpha$

$$\ddot{z} = \left(\frac{\alpha^2}{2mc^2} \right) \frac{1}{z^3}$$

$$\dot{p}_z = \frac{p_0^2}{mc^2 z^3} + gm$$

$\dot{p}_0 = 0 \Rightarrow p_0 = D$ (a constant)

$$\dot{z} = \frac{p_z}{m}$$

$$\dot{z} = \frac{p_0}{mc^2 z^2}$$

$$\ddot{z} = \frac{\dot{p}_z}{m} = \frac{p_0^2}{mc^2 z^3} + g$$

$$\ddot{z} = \frac{D}{mc^2} \left(\frac{1}{z^3} \right) + g$$

$$\ddot{z} = \frac{A}{z^3} + g$$

Multiplying by $2\dot{z} dt$

$$d[(\dot{z})^2] = \frac{2A\dot{z} dt}{z^3} + 2\dot{z}g dt$$

$$= 2A \frac{dz}{z^3} + 2gz$$

Integrating

$$(\dot{z})^2 = -\frac{A}{z^2} + 2gz + D_1$$

$$\frac{dz}{dt} = \sqrt{2gz - \frac{A}{z^2} + D_1}$$

$$\frac{z dz}{\sqrt{2gz^3 - A + D_1 z^2}} = dt$$

$$\ddot{z} = \frac{A}{z^3} + g$$