

NEXT IAS

(To be filled by candidate)

Name of Candidate: Kamishak Aggarwal

Roll No.: MT 24MATL1011 Date of Examination: 31/7

Mobile Number: 9771000000 E-mail: 31/7

Exam Centre: Old Rajinder Nagar ☐ Online ☒

CSE (MAINS) TEST SERIES-2024

Test <u>7</u>	MATHEMATICS
Dated : 14-07-2024	OPTIONAL

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS*(Please read each of the following instructions carefully before attempting questions)**There are EIGHT questions divided in Two sections and printed in ENGLISH.**Candidate has to attempt FIVE questions in all.**Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.**The number of marks carried by a question/part is indicated against it.**Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question cum Answer (QCA) booklet in the space provided.**No marks will be given for answers written in a medium other than the authorized one.**Word limit in questions, wherever specified, should be adhered to.**Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.***DELHI CENTRE:****Old Rajinder Nagar:** 27-B, Pusa Road, Metro Pillar no. 118, Near Karol Bagh Metro, New Delhi-110060 | Phone: 8081300200**Mukherjee Nagar:** 1422, Main Mukherjee Nagar Road, Near Balra Cinema, New Delhi-110009 | Phone: 8081300200**BHOPAL CENTRE:** Plot No. 46 Zone - 2, M.P. Nagar, Bhopal - 462011 | Phone: 8827664612, 8081300200**JAIPUR CENTRE:** Plot No. 6 & 7, 3rd Floor, Sree Gopal Nagar, Gopalpura Bypass, Jaipur - 302015 | Phone: 8358200511**PRAYAGRAJ CENTRE:** 31/31 Sardar Patel Marg, Civil Lines, Prayagraj, Uttar Pradesh - 211001 | Phone: 9968857757

(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	20			
2. (b)	18	15			
2. (c)	21	15			
3. (a)	24	15			
3. (b)	27	15			
3. (c)	30	20			
4. (a)	34	15			
4. (b)	37	20			
4. (c)	41	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	15			
6. (b)	57	20			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	15			
7. (c)	70	20			
8. (a)	74	20			
8. (b)	78	15			
8. (c)	81	15			
Grand Total					

Remarks

Observations:

1. The first part of the question is about the definition of a function. A function is a mapping from a set X to a set Y such that every element in X is mapped to exactly one element in Y .

2. The second part of the question is about the definition of a continuous function. A function $f: X \rightarrow Y$ is continuous if for every open set V in Y , the pre-image $f^{-1}(V)$ is an open set in X .

3. The third part of the question is about the definition of a compact set. A set K in a metric space is compact if every open cover of K has a finite subcover.

4. The fourth part of the question is about the definition of a closed set. A set C in a metric space is closed if it contains all its limit points.

5. The fifth part of the question is about the definition of a bounded set. A set S in a metric space is bounded if there exists a real number M such that $d(x, y) \leq M$ for all $x, y \in S$.

6. The sixth part of the question is about the definition of a complete metric space. A metric space (X, d) is complete if every Cauchy sequence in X converges to a limit in X .

7. The seventh part of the question is about the definition of a separable metric space. A metric space (X, d) is separable if it contains a countable dense subset.

8. The eighth part of the question is about the definition of a totally bounded set. A set S in a metric space is totally bounded if for every $\epsilon > 0$, there exists a finite number of points in S such that S is covered by the union of open balls of radius ϵ centered at these points.

9. The ninth part of the question is about the definition of a uniform continuity. A function $f: X \rightarrow Y$ is uniformly continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$ for all $x, y \in X$.

10. The tenth part of the question is about the definition of a Lipschitz continuous function. A function $f: X \rightarrow Y$ is Lipschitz continuous if there exists a constant L such that $d_Y(f(x), f(y)) \leq L d_X(x, y)$ for all $x, y \in X$.

11. The eleventh part of the question is about the definition of a differentiable function. A function $f: X \rightarrow Y$ is differentiable at a point $x \in X$ if there exists a linear map $df_x: X \rightarrow Y$ such that $d_Y(f(x+h), f(x)) = df_x(h) + o(\|h\|)$ as $h \rightarrow 0$.

12. The twelfth part of the question is about the definition of a Riemann integrable function. A function $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable if the upper and lower Riemann sums converge to the same value as the partition becomes finer.

SECTION-A

- Q.1 (a) Find a linear map $F: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ whose image is spanned by $(1, 2, 0, -4)$ and $(2, 0, -1, -3)$. (10)

without loss of generality
 $F(1, 0, 0) = (1, 2, 0, -4)$

$$F(0, 1, 0) = (2, 0, -1, -3)$$

Also $0 \in \mathbb{R}^4 \in \text{Image space}$

as $0 = 0(1, 2, 0, -4) + 0(2, 0, -1, -3) + 0(0, 0, 1)$

$$F(0, 0, 1) = (0, 0, 0, 0)$$

$$F(a, b, c) = F(a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1))$$

$$= \cancel{F(a)} a F(1, 0, 0) + b F(0, 1, 0) + c F(0, 0, 1)$$

by property of LT

$$= a[1, 2, 0, -4] + b[2, 0, -1, -3] + c[0, 0, 0, 0]$$

$$F(a, b) = [a+2b, 2a, -b, -4a-3b]$$

Q.1 (b)

Show that the vector $(1, 1, 2)$ is an eigen vector of the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$

corresponding to the eigen value 2.

(10)

$$\begin{aligned} Ax &= \lambda x \\ A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \lambda x \end{aligned}$$

where $\lambda = 2$

$$x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Given that $f(x+y) = f(x)f(y)$, $f(0) \neq 0$ for all real x, y and $f'(0) = 2$. Show that for all real x , $f'(x) = 2f(x)$. Hence find $f(x)$. (10)

$$\text{Let } y = 0$$

$$f(x) = f(0)f(x)$$

$$f(x)[1 - f(0)] = 0$$

$$f(x) = 0 \quad \text{or} \quad f(0) = 1$$

$$\text{If } f(x) = 0 \quad \forall x$$

$$\text{then } f'(x) = 0 \quad \forall x$$

$$\Rightarrow f'(0) = 0$$

which is not the case $\because f'(0) = 2$

$$\Rightarrow f(0) = 1$$

$$f(x-x) = f(x)f(-x) = 1$$

$$f(x)f(-x) = 1 \quad \text{--- (i)}$$

Differentiating w.r.t x

$$f'(x)f(-x) - f(x)f'(-x) = 0$$

$$\text{put } x = 0$$

$$f'(0)f(0) - f(0)f'(0) = 0$$

$$\text{Put } y = x$$

$$f(2x) = [f(x)]^2$$

$$2f'(2x) = 2f(x)f'(x)$$

~~4005~~
$$f'(2x) = f(x)f'(x) \quad \text{--- (i)}$$

Put $y = -2x$

$$f(-x) = f(x)f(-2x)$$

$$-f'(-x) = f'(x)f(-2x) - 2f'(-2x)f(x)$$

$$-f'(-x) = f'(x)f(-2x) - 2f'(-x)f(x)$$

put x as
 $-x$ in (ii)

Also Using (i)

$$-f'(-x) = f'(x)f(-2x) - 2f'(-x)$$

$$f'(-x) = f'(x)f(-2x)$$

$$f'(-x) = f'(x)f(-x)f(-x)$$

$$f'(x) = f'(-x)f(x)f(x)$$

$$f'(x) = f'(x)f(-x)f(x) \quad (\text{using ii})$$

$$f'(x) = f'(x) \quad \text{Using i}$$

Q.1 (d)

For all real number x , $f(x)$ is given as: $f(x) = \begin{cases} e^x + a \sin x, & x < 0 \\ b(x-1)^2 + x - 2, & x \geq 0 \end{cases}$. Find values of a and b for which f is differentiable at $x = 0$. (10)

Continuity is necessary for differentiability

$$\Rightarrow e^0 = b \cdot -2 \Rightarrow \cancel{b=0}$$

$$1 = b - 2 \Rightarrow b = 3$$

~~$f'(x) =$~~ ~~LH~~

$$\text{LHD} = \lim_{h \rightarrow 0^+} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \rightarrow 0^+} \frac{e^{-h} - a \sinh - 1}{-h}$$

$$= \lim_{h \rightarrow 0^+} \left(\frac{e^{-h} - 1}{-h} + \frac{a \sinh}{h} \right)$$

$$= 1 + a$$

$$\text{RHD} = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0^+} \frac{3(h-1)^2 + h - 2 - 1}{h}$$

Candidates
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$$\lim_{h \rightarrow 0^+} 3h - 5 = -5$$

$$\Rightarrow 1 + a = 5$$
$$a = 4$$

Find the locus of the point which moves so that its distance from the plane $x + y - z = 1$ is twice its distance from the line $x = -y = z$. (10)

let α, β, γ be the point
distance from plane

$$\left| \frac{\alpha + \beta - \gamma - 1}{\sqrt{1^2 + 1^2 + (-1)^2}} \right| = p$$

Distance from $\frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-0}{1}$

is x given by

$$x^2 (1^2 + (-1)^2 + 1^2) = [(-1)[\alpha] - \beta]^2 + (\beta + \gamma)^2 + (\gamma - \alpha)^2$$

$$p = 2x$$

$$p^2 = 4x^2$$

$$\frac{(\alpha + \beta - \gamma - 1)^2}{3} = \frac{4}{3} ((\alpha + \beta)^2 + (\beta + \gamma)^2 + (\gamma - \alpha)^2)$$

$$\cancel{7x^2 + 7p^2 + 9x^2}$$

$$4 \left[\cancel{(\alpha + \beta)^2} - \dots \right]$$

$$\alpha^2 + \beta^2 + \gamma^2 - 2(\alpha + \beta)(\gamma + 1) = 4 \left[(\alpha + \beta)^2 + (\beta + \gamma)^2 + (\alpha - \gamma)^2 \right]$$

$$7\alpha^2 + 7\beta^2 + 9\gamma^2 + 6\alpha\beta + 10\beta\gamma - 6\alpha\gamma - 1 - 2\gamma + 2\alpha + 2\beta = 0$$

locus.

$$7(x^2 + y^2) + 9z^2 + 6xy + 10yz - 6xz + 2(x + y - z) - 1 = 0.$$

Q.2 (a)

Determine an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix, where

$$A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$$

(20)

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)[(\lambda+8)^2 - 1] - 4[4(-8-\lambda) - 4]$$

$$-4[-4 - 4(8+\lambda)] = 0$$

$$(\lambda+8)^2(7-\lambda) + \lambda - 7 + 16(\lambda+8) + 16 + 16 + 16(\lambda+8) = 0$$

$$(\lambda+8)$$

$$(\lambda+8)[32 + (\lambda+8)(7-\lambda)] + 25 + \lambda = 0$$

$$(\lambda+8)[\cancel{32} - \lambda^2 - \lambda + 88] + 25 + \lambda = 0$$

$$(\lambda+8)(\lambda^2 + \lambda - 88) - 25 - \lambda = 0$$

$$\lambda^3 + 9\lambda^2 - 81\lambda - 729 = 0$$

$$(\lambda-9)(\lambda^2 + 18\lambda + 81) = 0$$

$$(\lambda - 9)(\lambda + 9)^2 = 0$$

$$\lambda = 9 \Rightarrow \lambda = 9, -9, -9.$$

$$(A - \lambda I) X = 0$$

$$\begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 2R_1; R_3 \rightarrow R_3 - 2R_1$$

$$\begin{bmatrix} -2 & 4 & -4 \\ 0 & -9 & -9 \\ 0 & -9 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 & -4 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{let } x_3 = k$$

$$-2x_1 + 4x_2 - 4x_3 = 0$$

$$-9x_2 - 9x_3 = 0$$

$$\text{let } x_3 = k$$

$$x_2 = -k$$

$$-2x_1 - 4k - 4k = 0$$

$$x_1 = -4k.$$

$$(x_1 \ x_2 \ x_3)^T = \begin{pmatrix} 4 & -1 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & 1/4 & -1/4 \end{pmatrix}$$

for $\lambda = -9$

$$\begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$R_2 \rightarrow R_2 - R_1/4$$

$$\begin{bmatrix} 16 & 4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{let } x_2 = k_1 ; \quad x_3 = k_2$$

$$\Rightarrow 16x_1 + 4x_2 - 4x_3 = 0$$

$$16x_1 + 4k_1 - 4k_2 = 0$$

$$x_1 = -\frac{1}{4}k_1 + \frac{1}{4}k_2$$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}k_1 + \frac{1}{4}k_2 \\ k_1 \\ k_2 \end{bmatrix}$$

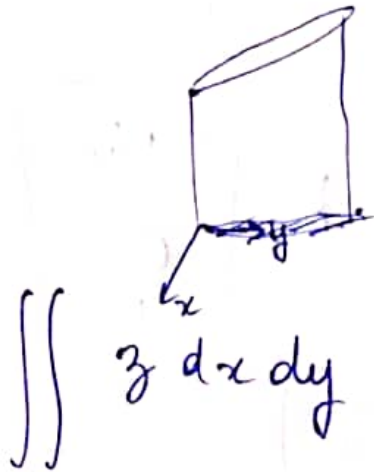
$$= k_1 \begin{bmatrix} -1/4 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1/4 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1/4 & 1/4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$PP^T = I$$

also $P^{-1}AP = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$

Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and planes $y + z = 4$ and $z = 0$.
(15)



$$\iint z \, dx \, dy$$

$$= \iint_R (4 - y) \, dx \, dy$$

R is ~~the~~ $x^2 + y^2 = 4$; $z = 0$

let $x = r \cos \theta$

$y = r \sin \theta$

converting to polar.

$$\int_{\theta=0}^{2\pi} \int_{r=0}^2 (4 - r \sin \theta) r \, dr \, d\theta$$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^2 (4r - r^2 \sin \theta) \, dr \, d\theta$$

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$$= \int_{\theta=0}^{2\pi} \left(2r^2 - \frac{r^3}{3} \sin \theta \right)^2 d\theta$$

$$= \int_0^{2\pi} 8 - \frac{8}{3} \sin \theta d\theta$$

$$16\pi + \frac{8}{3}(0) = \boxed{16\pi}$$



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A line with DR's $(7, -5, 2)$ is drawn to intersect the lines $\frac{x-7}{1} = \frac{y+2}{1} = \frac{z-5}{3}$, $\frac{x-3}{2} = \frac{y-5}{4} = \frac{z+3}{-3}$. Find the coordinates of the points of intersection and the length intercepted on it. (15)

let the line L be

$$\frac{x-\alpha}{7} = \frac{y-\beta}{-5} = \frac{z-\gamma}{2} = r_1$$

intersect $\frac{x-7}{-1} = \frac{y+2}{1} = \frac{z-5}{3}$

$$x = 7r_1 + \alpha; \quad y = \beta - 5r_1; \quad z = 2r_1 + \gamma$$

satisfy

$$\frac{7r_1 + \alpha - 7}{-1} = \beta - 5r_1 + 2 = \frac{2r_1 + \gamma - 5}{3}$$

$$\alpha + \beta + 12r_1 = 5 \quad \text{---i}$$

$$\gamma - 3\beta + 17r_1 = 11 \quad \text{---ii}$$

L_2 intersect $\frac{x-3}{2} = \frac{y-5}{4} = \frac{z+3}{-3}$

$$\frac{7r_2 + \alpha - 3}{2} = \frac{\beta - 5r_2 - 5}{4} = \frac{2r_2 + \gamma + 3}{-3}$$

$$19r_2 + 2\alpha - \beta = 1 \quad \text{---iii}$$

$$7r_2 - 3\beta - 4\gamma = 27 \quad \text{---iv}$$

let $(a_1, a_2, a_3); (b_1, b_2, b_3)$ be the intersection coordinates respectively

$$\Rightarrow \frac{a_1 - 7}{-1} = \frac{a_2 + 2}{1} = \frac{a_3 - 5}{3}$$

$$\frac{b_1 - 3}{2} = \frac{b_2 - 5}{4} = \frac{b_3 + 3}{-3}$$

Also $\frac{a_1 - b_1}{7} = \frac{a_2 - b_2}{-5} = \frac{a_3 - b_3}{2}$

(DR's of line joining (a_1, a_2, a_3) and (b_1, b_2, b_3) are proportional to $7, -5, 2$).

$$a_1 + a_2 = 5 \Rightarrow a_1 = 5 - a_2$$

$$3a_2 - a_3 = -11 \Rightarrow a_3 = 3a_2 + 11$$

$$2b_1 - b_2 = 1 \Rightarrow b_2 = 2b_1 - 1$$

$$2b_3 + 3b_1 = 3 \Rightarrow b_3 = \frac{3 - 3b_1}{2}$$

$$5(a_1 - b_1) = 7(b_2 - a_2)$$

$$\Rightarrow 5(5 - a_2 - b_1) = 7(2b_1 - 1 - a_2)$$

$$\Rightarrow 32 + 2a_2 = 19b_1$$

$$2(a_1 - b_1) = 7(a_3 - b_3)$$

$$2(5 - a_2 - b_1) = 7\left(3a_2 + 11 - \frac{3 - 3b_1}{2}\right)$$

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$$4(5 - a_2 - b_1) = 7(6a_2 + 2a - 3 + 3b_1)$$

$$+38a_2 + 113 = -25b_1$$

$$a_2 = -\frac{421}{132}; b_1 = \frac{89}{66}$$

$$a_1 = \frac{1081}{132}$$

$$b_2 = \frac{56}{33}$$

$$a_3 = \frac{63}{44}$$

$$b_3 = -\frac{23}{44}$$

$$\left(\frac{1081}{132}, -\frac{421}{132}, \frac{189}{132}\right); \left(\frac{178}{132}, \frac{224}{132}, -\frac{69}{132}\right)$$

$$903, -645, 258$$

$$\text{length} = \frac{43578}{44}$$

Q.4 (a)

Consider the vector space $V(R)$ of all 2×2 matrices over the field R of real numbers. Let T be linear transformation on V that sends each matrix X onto

AX , where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find the matrix of T with respect to the ordered basis

$$B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \text{ for } V \text{ where } \alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(15)

$$A\alpha_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \alpha_1 + \alpha_3$$

$$A\alpha_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \alpha_2 + \alpha_4$$

$$A\alpha_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \alpha_1 + \alpha_3$$

$$A\alpha_4 = \alpha_2 + \alpha_4 \Rightarrow \text{coordinate vector} = (0 \ 1 \ 0 \ 1)^T$$

Writing coordinate vectors columns wise

$$[T]_B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Find constant a and b for which $F(a, b) = \int_0^\pi \{\sin x - (ax^2 + bx)\}^2 dx$ is a minimum.

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(20)

$$\begin{aligned}
 F(a, b) &= \int_0^\pi \{\sin x - (ax^2 + bx)\}^2 dx \\
 &= \int_0^\pi \sin^2 x + (ax^2 + bx)^2 - 2(ax^2 + bx)\sin x \, dx \\
 &= \int_0^\pi \frac{1 - \cos 2x}{2} + a^2 x^4 + b^2 x^2 + 2abx^3 - 2(ax^2 + bx)\sin x \, dx \\
 &= \frac{\pi}{2} + \frac{a^2}{5}\pi^5 + \frac{b^2}{3}\pi^3 + \frac{ab}{2}\pi^4 - 2 \left[(ax^2 + bx)(-\cos x) - (2ax + b)(-\sin x) \right]_0^\pi \\
 &= \frac{\pi}{2} + \frac{a^2}{5}\pi^5 + \frac{b^2}{3}\pi^3 + \frac{ab}{2}\pi^4 + \frac{4a}{2}\pi^2 - 2(a\pi^2 + b\pi) \\
 &= F(a, b)
 \end{aligned}$$

$$\frac{\partial F}{\partial a} = \frac{2a}{5}\pi^5 + \frac{\pi^4}{2}b - 4a\pi^2 = 0 \quad \text{--- i}$$

$$\frac{\partial F}{\partial b} = \frac{2b}{3}\pi^3 + \frac{a}{2}\pi^4 - 2\pi = 0 \quad \text{--- ii}$$

$$4a\pi^3 + 5b\pi^2 = 2$$

$$3a\pi^3 + 4b\pi^2 = 2$$

$$a\pi^3 + b\pi^2 = 0$$

$$b = -a\pi$$

$$4a\pi^3 - 5a\pi^3 = 2 \Rightarrow a = -\frac{2}{\pi^3}$$

$$b = \frac{2}{\pi^2}$$

$$\frac{\partial^2 F}{\partial a^2} = \frac{2\pi^5}{\pi^5} = 4$$

$$\frac{\partial^2 F}{\partial b^2} = \frac{2\pi^3}{3} = 8$$

$$\frac{\partial^2 F}{\partial a \partial b} = \frac{\pi^4}{2} = 8$$

 $\partial^2 F$

$$4t - 8^2 =$$

$$\left(\frac{4}{15} - \frac{1}{4} \right) \pi^8$$

$$\frac{1}{16} \pi^8 > 0$$

$$4 > 0, 8 > 0$$

 $\Rightarrow$ Minima

$$a = \frac{-2}{\pi^3}$$

$$b = \frac{2}{\pi^2}$$

Prove that in general two generators of the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ can be drawn to cut a given generator at right angles.

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2} \quad (15)$$

$$\frac{x}{a} - \frac{z}{c} = \lambda \left(1 - \frac{y}{b}\right); \quad \frac{x}{a} + \frac{z}{c} = \frac{1}{\lambda} \left(1 + \frac{y}{b}\right)$$

$$\frac{x}{a} + \frac{z}{c} = \mu \left(1 - \frac{y}{b}\right); \quad \frac{x}{a} - \frac{z}{c} = \frac{1}{\mu} \left(1 + \frac{y}{b}\right)$$

be two system of generator.

let λ system is given.

let l_1, m_1, n_1 be dcs of that generator.

$$\frac{l_1}{a} + \frac{m_1}{b} \lambda - \frac{n_1}{c} = 0$$

$$\frac{l_1}{a} - \frac{m_1}{b\lambda} + \frac{n_1}{c} = 0$$

$$\frac{l_1}{\frac{\lambda}{bc} - \frac{1}{bc\lambda}} = \frac{m_1}{-\frac{z}{ac}} = \frac{n_1}{-\frac{1}{ab\lambda} - \frac{\lambda}{ab}}$$

Now this is similar the dcs l_2, m_2, n_2 of μ system are.

$$\frac{l_2}{-\frac{\mu}{bc} + \frac{1}{bc\mu}} = \frac{m_2}{\frac{z}{ac}} = \frac{n_2}{\frac{1}{ab\mu} - \frac{\mu}{ab}}$$

These 2 are $\perp$

$$\Rightarrow \vec{a} \cdot \vec{b} = 0$$

i.e.

$$\left(\frac{\lambda}{bc} - \frac{1}{bc\lambda} \right) \left(\frac{\mu}{bc} + \frac{1}{bc\mu} \right) - \frac{4}{(ac)^2}$$

$$+ \left(\frac{1}{ab\lambda} + \frac{\lambda}{ab} \right) \left(\frac{1}{ab\mu} + \frac{\mu}{ab} \right) = 0$$

$$\frac{(\lambda^2 - 1)(1 - \mu^2)}{(bc)^2 \lambda \mu} - \frac{4}{(ac)^2} + \frac{(\lambda^2 + 1)(\mu^2 + 1)}{(ab)^2 \mu \lambda} = 0$$

$$(\lambda^2 - 1)(1 - \mu^2)(a^2) - 4b^2 \lambda \mu + (\lambda^2 + 1)(\mu^2 + 1)c^2 = 0$$

$$\mu^2 (-\lambda^2 a^2 + c^2(1 + \lambda^2)) - 4b^2 \lambda \mu + (\lambda^2 + 1)c^2 + a^2(\lambda^2 - 1) = 0$$

This is quadratic in μ
hence two values of μ
possible for given λ

$\therefore$ 2 generator perpendicular to
one given generator possible

Q.5 (a)

Given that $L\{f(t); p\} = F(p)$. Show that $\int_0^\infty \frac{f(t)}{t} dt = \int_0^\infty F(x) dx$. Hence evaluate the

integral $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$.

(10)

$$I = \int_0^\infty e^{-st} \frac{f(t)}{t} dt$$

given $\int_0^\infty e^{-st} f(t) dt = F(s)$

$$L(f'(t)) = sF(s) - f(0)$$

$$\int_0^\infty e^{-st} f'(t) dt = sF(s) - f(0)$$

$$RHS = \int_{s=0}^\infty \int_{t=0}^\infty e^{-st} f(t) dt ds$$

$$= \int_{t=0}^\infty \int_{s=0}^\infty e^{-st} f(t) ds dt$$

$$= \int_{t=0}^\infty \frac{f(t)}{t} [0 - 1] dt$$

$$= \int_0^\infty \frac{f(t)}{t} dt$$

Q.5 (b)

Evaluate $\int_{0,0}^{2,1} (10x^4 - 2xy^3) dx - 3x^2y^2 dy$ along the path $x^4 - 6xy^3 = 4y^2$.

(10)

$$x^4 - 6xy^3 = 4y^2$$

$$\int M dx + N dy$$

$$M = 10x^4 - 2xy^3$$

$$N = -3x^2y^2$$

$$\frac{\partial M}{\partial y} = -6xy^2 = \frac{\partial N}{\partial x}$$

$\Rightarrow$ It is exact

Hence path independent

$$\int M dx + N dy$$

$$\int_{y \text{ const}} M dx + \int N dy$$

term not having x

$$\int (10x^4 - 2xy^3) dx + \int 0 dy$$

$$\left[2x^5 - x^2y^3 \right]_{(0,0)}^{(2,1)}$$

$$= 60$$

Don't How to do without exact parameter is difficult here



$(-2, 1)$ is a point on the curve

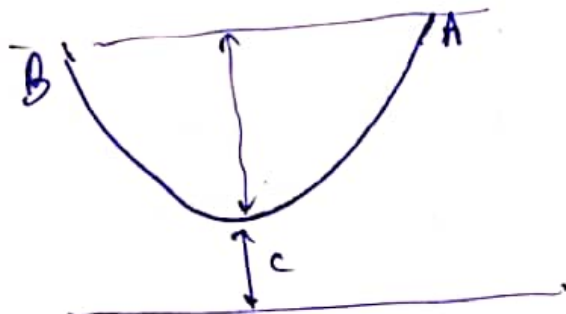
Let the equation of the curve be $y = ax^2 + bx + c$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1, R_3 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & -4 & 0 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A uniform chain of length l , which can just bear a tension of n times its weight, is stretched between two points on the same horizontal line. Show that the

least possible sag in the middle is $l \left\{ n - \sqrt{n^2 - \frac{1}{4}} \right\}$. (10)



Sag will be least when T_A is maximum

$$T_A = nwl = wy_A$$

$$y_A = nl$$

$$T_0 = wc$$

$$T_A^2 = T_0^2 + \left(\frac{wl}{2} \right)^2$$

$$T_0^2 = (wl)^2 \left[n^2 - \frac{1}{4} \right]$$

$$c^2 = l^2 \left[n^2 - \frac{1}{4} \right]$$

$$\text{Sag} = y_A - c$$

$$= nl - l \sqrt{n^2 - \frac{1}{4}}$$

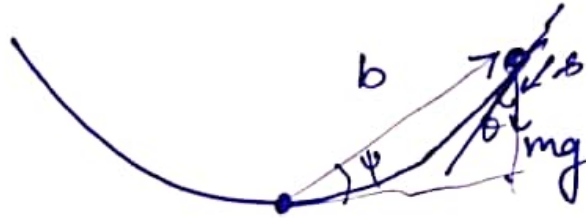
$$= l \left\{ n - \sqrt{n^2 - \frac{1}{4}} \right\}$$

Q.5 (d) State Kepler's laws of Planetary Motion.

(10)

- i) Planet revolves around Sun which is at focus.
- ii) Sectoral area swept in unit time by planet and radius vector is constant.
- iii) Time period's square is directly proportional to cube of semi major axis.

If a particle slides down a smooth cycloid, starting from a point whose actual distance from the vertex is b . Prove that its speed at any time t is $\frac{2\pi b}{T} \sin\left(\frac{2\pi t}{T}\right)$, where T is the time of complete oscillation of the particle. (10)



$$\psi = 2\theta$$

$$-\frac{d^2 s}{dt^2} = mg \cos \theta$$

$$-v \frac{dv}{ds} = mg \cos \theta$$

$$s^2 = 8ay$$

$$x = a(\theta + \sin \theta)$$

$$y = a(1 - \cos \theta)$$

$$2s ds = 8a dy$$

$$\text{also } \cos \theta = 1 - y/a$$

$$-\frac{v dv}{8a dy} (2s) = mg \left(1 - \frac{y}{a}\right)$$

$$-\frac{v dv}{4a} = mg \left(1 - \frac{y}{a}\right) \frac{dy}{\sqrt{8ay}}$$

$$\cancel{-\frac{v^2}{8a}} - \frac{v dv}{4a} = \frac{mg}{\sqrt{8a}} \left(y^{-1/2} - \frac{y^{1/2}}{a} \right) dy$$

$$\left[-\frac{v^2}{8a} \right] = \frac{mg}{\sqrt{8a}} \left(2\sqrt{y} - \frac{2}{3} \frac{y^{3/2}}{a} \right) + C_1$$

[Handwritten signature]



Q.7 (a)

Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$; $a > b > 0$ are constants, and λ is a parameter. Show that the given family of curves is self-orthogonal. (15)

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$$

Differentiating w.r.t x

$$\frac{2x}{a^2 + \lambda} + \frac{2y y'}{b^2 + \lambda} = 0$$

$$\Rightarrow (b^2 + \lambda)(x) + y y'(a^2 + \lambda) = 0$$

$$\Rightarrow \lambda [x + y y'] + a^2 y y' + b^2 x = 0$$

$$\Rightarrow \lambda = - \frac{(a^2 y y' + b^2 x)}{x + y y'}$$

$$a^2 + \lambda = \frac{x(a^2 - b^2)}{x + y y'}$$

$$b^2 + \lambda = \frac{y y'(b^2 - a^2)}{x + y y'}$$

$$\frac{x^2 (x + y y')}{x(a^2 - b^2)} + \frac{y^2 (x + y y')}{y y'(b^2 - a^2)} = 1$$

$$x y'(x + y y') - y(x + y y') = a^2 - b^2$$

(1)

Is the differential equation
of given confocal curve.

Replacing y' by $-1/y'$

$$\frac{x^2 \left(x - \frac{y}{y'} \right)}{x(a^2 - b^2)} + \frac{y(-y') \left(x - \frac{y}{y'} \right)}{\cancel{(-y')} (b^2 - a^2)} = 1$$

$$\frac{x(xy' - y)}{(a^2 - b^2)y'} + \frac{yy'(xy' - y)}{(a^2 - b^2)} = 1$$

~~xy~~ $x(xy' - y) + yy'(xy' - y)$

$$xy'(x + yy') - y(x + yy') = a^2 - b^2$$

which is same as (i)

Hence given family is
~~self~~ orthogonal.

Candidates
must not write
on this margin

(b) If $\vec{F} = \left(y \frac{\partial \phi}{\partial z} - z \frac{\partial \phi}{\partial y} \right) \hat{i} + \left(z \frac{\partial \phi}{\partial x} - x \frac{\partial \phi}{\partial z} \right) \hat{j} + \left(x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \right) \hat{k}$, then prove that

$$\vec{F} - (\vec{r} \times \nabla \phi) = \vec{F} \cdot \vec{r} = \vec{F} \cdot \nabla \phi = 0.$$

(15)

$$\vec{r} \times \nabla \phi = (x\hat{i} + y\hat{j} + z\hat{k}) \times \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i} \left(y \frac{\partial \phi}{\partial z} - z \frac{\partial \phi}{\partial y} \right) + \hat{j} \left(z \frac{\partial \phi}{\partial x} - x \frac{\partial \phi}{\partial z} \right) + \hat{k} \left(x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \right) = \vec{F}$$

$$\vec{F} - \vec{r} \times \nabla \phi = 0.$$

$$\vec{F} \cdot \vec{r} = \sum x \left(y \frac{\partial \phi}{\partial z} - z \frac{\partial \phi}{\partial y} \right)$$

$$= \left[xy \frac{\partial \phi}{\partial z} - xz \frac{\partial \phi}{\partial y} + yz \frac{\partial \phi}{\partial x} - xy \frac{\partial \phi}{\partial z} + xz \frac{\partial \phi}{\partial y} - zy \frac{\partial \phi}{\partial x} \right] = 0$$

$$\vec{F} \cdot \nabla \phi = \frac{\partial \phi}{\partial x} \left(y \frac{\partial \phi}{\partial z} - z \frac{\partial \phi}{\partial y} \right) + \frac{\partial \phi}{\partial y} \left(z \frac{\partial \phi}{\partial x} - x \frac{\partial \phi}{\partial z} \right) + \frac{\partial \phi}{\partial z} \left(x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \right) = 0.$$

or

$$\vec{F} = \vec{E} \times \nabla \phi$$

$$\vec{F} \cdot \vec{E} = \vec{E} \cdot (\vec{E} \times \nabla \phi) = 0$$

$$\vec{F} \cdot \nabla \phi = \nabla \phi \cdot (\vec{E} \times \nabla \phi) = 0$$

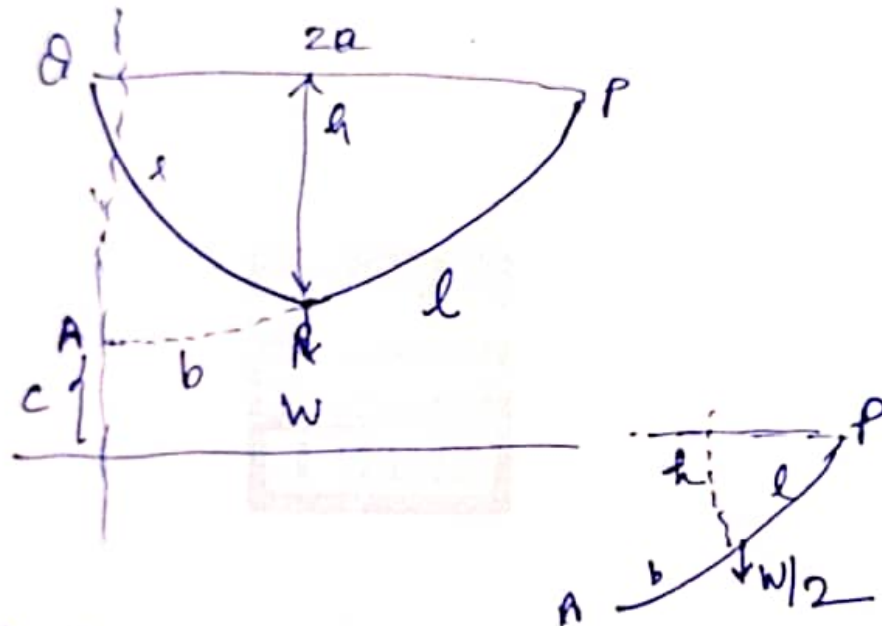
$$\therefore (\vec{a} \cdot (\vec{a} \times \vec{b}) = 0)$$

Q.7 (c)

PR and QR are two heavy strings tied together at R and carrying a weight W at R. P and Q are two points in the same horizontal line and $2a$ is the distance between them. l is the length of each string and h is the depth of R below PQ. Prove that

$$(i) \quad l^2 - h^2 = 2c^2 \left(\cosh \frac{a}{c} - 1 \right)$$

(ii) Tension at P or Q = $\frac{1}{2h} \{ W + (l^2 + h^2)w \}$, where c , is the parameter of the catenary and w is the line density of the string. (20)



AP is catenary.

$$x_P - x_R = a$$

$$y_P - y_R = h$$

$$x_P = c \cosh \frac{x}{c}$$

$$s_P - s_R = l$$

$$s = c \sinh \frac{x}{c}$$

$$y = c \cosh \frac{x}{c}$$

$$y^2 = c^2 + s^2$$

$$y_P^2 - y_R^2 = s_P^2 - s_R^2$$

$$h(y_P + y_R) = l(s_P + s_R)$$

$$y_p = c \cosh \frac{x_p}{c}$$

$$y_R = c \cosh \frac{x_R}{c}$$

$$y_R = c \cosh \frac{x_R + a}{c}$$

$$s_R = c \sinh \frac{x_R}{c}$$

$$s_p = c \sinh \frac{x_R + a}{c}$$

$$s_p^2 - y_R^2 = c \left[\sinh^2 \frac{x_R + a}{c} - \cosh^2 \frac{x_R + a}{c} \right]$$

$$= c \left[\left[\frac{e^{\frac{x_R + a}{c}} - e^{-\frac{x_R + a}{c}}}{2} \right]^2 - \left[\frac{e^{\frac{x_R + a}{c}} + e^{-\frac{x_R + a}{c}}}{2} \right]^2 \right]$$

$$= c [1]$$

$$s_p^2 - y_p^2 = c$$

$$s_R^2 - y_R^2$$

$$s_R^2 - y_R^2 = (s_p - s_R)^2 - (y_p - y_R)^2$$

$$= \left[c^2 \left\{ \sinh^2 \frac{x_R + a}{c} - \sinh^2 \frac{x_R}{c} \right\} - \left\{ \cosh^2 \frac{x_R + a}{c} - \cosh^2 \frac{x_R}{c} \right\} \right]$$

$$= c^2 \left[2 \sinh \frac{x_R + a}{c} \cosh \frac{x_R}{c} - 2 \cosh \frac{x_R + a}{c} \sinh \frac{x_R}{c} \right]$$

$$= 2c^2 \left[-1 + \cosh \left[\frac{x_R + a}{c} - \frac{x_R}{c} \right] \right] = 2c^2 \left[\cosh \frac{a}{c} - 1 \right]$$

$$ii) T_p = \frac{1}{2h} \left\{ lw + (l^2 + h^2)w \right\}$$

~~$$w + wb =$$~~

$$T = wb = \frac{w}{2} \quad (T = wy)$$

$$\Rightarrow b = \frac{w}{2w}$$

$$T_p = y_p w$$

$$y_p = y_R + h$$

$$s^2 + c^2 = y^2$$

$$y_R^2 = b^2 + c^2$$

$$y_R = \sqrt{b^2 + c^2}$$

$$y_p = \sqrt{b^2 + c^2} + h$$

$$y_p^2 = c^2 + s^2$$

$$h^2 + b^2 + \cancel{c^2} + 2h\sqrt{b^2 + c^2} = \cancel{c^2} + (b+h)^2$$

$$2h\sqrt{b^2 + c^2} = l^2 + 2bl - h^2$$

$$4h^2(b^2 + c^2) = (l^2 + 2bl - h^2)^2$$

$$c^2 = \frac{(l^2 + 2bl - h^2)^2 - 4h^2b^2}{4h^2}$$

$$\frac{abc}{\sqrt{b^2+c^2}} = \frac{l^2 + 2bl - h^2}{2h}$$

$$y_p = h + \frac{l^2 + 2bl - h^2}{2h}$$

$$y_p = \frac{h^2 + l^2 + 2bl}{2h}$$

$$T = wy_p = \frac{w}{2h} \left[(l^2 + h^2) + \cancel{2b} \frac{w}{2w} l \right]$$

$$= \frac{1}{2h} \left[w(l^2 + h^2) + wl \right]$$