

NEXT IAS

(To be filled by candidate)

Name : ANNU KOUSIK NARASIMHARol No. : OAC24MTTA003 Date of Exam. : 05-08-2024Mobile No. : XXXXXXXXXX

Exam Centre : Old Rajinder Nagar ☐ Bhopal ☐ Jaipur ☐
 Mukherjee Nagar ☐ Prayagraj ☐ Online ☒

CSE (MAINS) TEST SERIES-2024

Test-07	MATHEMATICS
Dated : 04-08-2024	OPTIONAL

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

*(Please read each of the following instructions carefully before attempting questions)**There are EIGHT questions divided in Two sections and printed in ENGLISH.**Candidate has to attempt FIVE questions in all.**Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.**The number of marks carried by a question/part is indicated against it.**Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.**No marks will be given for answers written in a medium other than the authorized one.**Word limit in questions, wherever specified, should be adhered to.**Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.***DELHI CENTRE:**

Old Rajinder Nagar: 27-B, Pusa Road, Metro Pillar no. 118, Near Karol Bagh Metro, New Delhi-110060 | Ph: 8081300200
 Mukherjee Nagar: 1422, Main Mukherjee Nagar Road, Near Batra Cinema, New Delhi-110009 | Ph: 8081300200

BHOPAL CENTRE: Plot No. 46 Zone - 2, M.P Nagar, Bhopal - 462011 | Ph: 8827664612, 8081300200**JAIPUR CENTRE:** 6 & 7, 3rd Floor, Sree Gopal Nagar, Gopalpura Bypass, Jaipur - 302015 | Phone: 9358200511**PRAYAGRAJ CENTRE:** 31/31 Sardar Patel Marg, Civil Lines, Prayagraj, Uttar Pradesh - 211001 | Phone: 9958857757

(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	20			
2. (b)	18	15			
2. (c)	21	15			
3. (a)	24	15			
3. (b)	27	15			
3. (c)	30	20			
4. (a)	34	15			
4. (b)	37	20			
4. (c)	41	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	15			
6. (b)	57	20			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	15			
7. (c)	70	20			
8. (a)	74	20			
8. (b)	78	15			
8. (c)	81	15			
Grand Total					

Remarks

Observations:

The following is a list of observations
 made during the experiment.
 (1) The first observation is that
 the temperature of the water
 (2) The second observation is that
 the temperature of the water
 (3) The third observation is that
 the temperature of the water
 (4) The fourth observation is that
 the temperature of the water
 (5) The fifth observation is that
 the temperature of the water
 (6) The sixth observation is that
 the temperature of the water
 (7) The seventh observation is that
 the temperature of the water
 (8) The eighth observation is that
 the temperature of the water
 (9) The ninth observation is that
 the temperature of the water
 (10) The tenth observation is that
 the temperature of the water

SECTION-A

- Q.1 (a) Find a linear map $F: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ whose image is spanned by $(1, 2, 0, -4)$ and $(2, 0, -1, -3)$. (10)

Finding the image i.e. spanning set of
 $(1, 2, 0, -4), (2, 0, -1, -3)$

$$\begin{aligned} \text{Image set} &= a(1, 2, 0, -4) + b(2, 0, -1, -3) \\ &= (a+2b, 2a, -b, -4a-3b) \end{aligned}$$

Let the element in domain set be
 $(a, b, c) \in \mathbb{R}^3$.

By inspection, the linear map is

$$F(a, b, c) = (a+2b, 2a, -b, -4a-3b)$$

is the reqd linear map.

Q.1 (b)

Show that the vector $(1, 1, 2)$ is an eigen vector of the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ corresponding to the eigen value 2. (10)

Let $X = (1, 1, 2)^T$ and $\lambda = 2$.

Given $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$. The vector is

eigen vector if $AX = \lambda X$

$$\lambda X = 2 \times \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \rightarrow \textcircled{1}$$

$$AX = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+1-2 \\ 2+2-2 \\ 2+2+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \rightarrow \textcircled{2}$$

$$\textcircled{1} = \textcircled{2}$$

$\therefore AX = \lambda X \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is the eigen vector of A corresponding to $\lambda = 2$.

Q.1 (c)

Given that $f(x+y) = f(x)f(y)$, $f(0) \neq 0$ for all real x, y and $f'(0) = 2$. Show that for all real x , $f'(x) = 2f(x)$. Hence find $f(x)$. (10)

$$f(x+y) = f(x)f(y) ; f(0) \neq 0$$

$$x=y=0 \Rightarrow f(0) = [f(0)]^2 \Rightarrow f(0) = 1 \quad (\because f(0) \neq 0)$$

$$f'(x+y) = f'(x)f(y) + f'(y)f(x)$$

$$\text{For } x=0=y, \quad f'(0) = 2f'(0)f(0) = 2f'(0)$$

~~$$\text{Let } x=x \text{ and } y=0$$~~

~~$$\Rightarrow f'(x) = f'(x)f(0) + f(0)f'(x)$$~~

$$\text{Let } y=x \Rightarrow f(2x) = (f(x))^2$$

$$\Rightarrow f(x) = (f(x/2))^2 \quad \text{--- (1)}$$

$$\text{Also } f'(x) = 2f(x)f'(x)$$

$$\Rightarrow f'(x) = 2f(x/2)f'(x/2) \quad \text{--- (2)}$$

~~Ans~~

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on this margin



- Q.1 (d) For all real number x , $f(x)$ is given as: $f(x) = \begin{cases} e^x + a \sin x, & x < 0 \\ b(x-1)^2 + x - 2, & x \geq 0 \end{cases}$. Find values of a and b for which f is differentiable at $x=0$. (10)

Checking continuity at $x=0$ ($\because$ continuity is essential for differentiability)

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} e^x + a \sin x = 1 \quad \text{--- (1)}$$

$$(\because \lim_{x \rightarrow 0^-} e^x = 1)$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} b(x-1)^2 + x - 2$$

$$= b(-1)^2 + 0 - 2 = b - 2 \quad \text{--- (2)}$$

$$\textcircled{1} = \textcircled{2} \Rightarrow b - 2 = 1 \Rightarrow \boxed{b = 3}$$

Checking differentiability at $x=0$:

$$f(0) = 3(0-1)^2 + 0 - 2 = \underline{1}$$

For differentiability,

$$\lim_{h \rightarrow 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h}$$

$$\lim_{h \rightarrow 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{e^h + a \sin h - 1}{h}$$

$$= \lim_{h \rightarrow 0^-} \frac{a \sinh}{h} = a \left(\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right) \text{--- (3)}$$

$$\lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h - 0} = \lim_{h \rightarrow 0^+} \frac{b(h-1)^2 + h - 2 - 1}{h}$$

$$= \lim_{h \rightarrow 0^+} \frac{b + h - 3}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \text{--- (4)}$$

($\because b = 3$)

$$\textcircled{3} = \textcircled{4}$$

$$\Rightarrow \boxed{a = 1}$$

$\therefore$ The values of a and b are

$$\boxed{a = 1, b = 3}$$

- Q.1 (e) Find the locus of the point which moves so that its distance from the plane $x + y - z = 1$ is twice its distance from the line $x = -y = z$. (10)

Let the reqd point be (x_1, y_1, z_1)

distance of the point from the plane

$$\text{is } d_1 = \frac{|x_1 + y_1 - z_1 - 1|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{|x_1 + y_1 - z_1 - 1|}{\sqrt{3}}$$

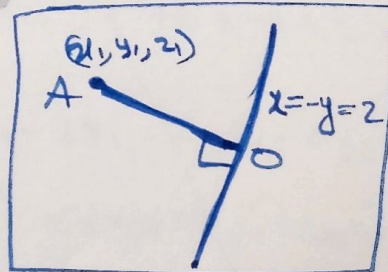
Finding shortest distance from the line:

Parametric pt = $(t, -t, t)$

dis of the point from line = $(x_1 - t, y_1 + t, z_1 - t)$

the line segment AO
is perpendicular to line

$$\Rightarrow 1(x_1 - t) - 1(y_1 + t) + 1(z_1 - t) = 0$$



$$\Rightarrow (x_1 - y_1 + z_1) - 3t = 0 \Rightarrow t = \frac{(x_1 - y_1 + z_1)}{3}$$

$\Rightarrow$ the pt on the line is $\left(\frac{x_1 - y_1 + z_1}{3}, \frac{-x_1 + y_1 - z_1}{3}, \frac{x_1 - y_1 + z_1}{3} \right)$

The distance from line $\Rightarrow d_2$ is

$$\sqrt{\left(x_1 - \frac{x_1 - y_1 + z_1}{3}\right)^2 + \left(y_1 + \frac{x_1 - y_1 + z_1}{3}\right)^2 + \left(z_1 - \frac{x_1 - y_1 + z_1}{3}\right)^2}$$

$$\Rightarrow d_2 = \sqrt{\frac{(2x_1 + y_1 - 2)^2}{9} + \left(\frac{x_1 + 2y_1 + 2}{9}\right)^2 + \left(\frac{-x_1 + y_1 + 2z_1}{3}\right)^2}$$

Now given that $d_1 = 2d_2$

$$\Rightarrow \frac{|x_1 + y_1 - 2|}{\sqrt{3}} = 2 \sqrt{\frac{(2x_1 + y_1 - 2)^2}{9} + \frac{(x_1 + 2y_1 + 2)^2}{9} + \frac{(x_1 + y_1 + 2z_1)^2}{9}}$$

$$\Rightarrow \frac{(x_1 + y_1 - 2)^2}{3} = \frac{4}{9} (6x_1^2 + 6y_1^2 + 6z_1^2 + 4x_1y_1 - 4x_1z_1 + 6y_1z_1)$$

$$\Rightarrow 3(x_1^2 + y_1^2 + z_1^2 + 2x_1y_1 - 2x_1z_1 - 2y_1z_1 + 2z_1) = 24x_1^2 + 24y_1^2 + 24z_1^2 + 16x_1y_1 - 16x_1z_1 + 24y_1z_1$$

$$\Rightarrow 21x_1^2 + 21y_1^2 + 21z_1^2 + 10x_1y_1 - 10x_1z_1 + 30y_1z_1 + 6x_1 + 6y_1 - 6z_1 - 3 = 0.$$

$\therefore$ The locus of the pt is

$$21(x^2 + y^2 + z^2) + 10xy - 10xz + 30yz + 6x + 6y - 6z - 3 = 0$$

Q.2 (a)

Determine an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix, where

$$A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$$

(20)

Finding the eigen values and eigen vectors of A .

$$\Rightarrow |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - (-9)\lambda^2 + (-81)\lambda - 729 = 0$$

Solving, we get roots = $9, -9, -9$

Finding the eigen vectors:

① $\lambda = 9$. Let $X_1 = (x_1, y_1, z_1)^T$

$$AX_1 = \lambda X_1 \Rightarrow \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 9x_1 \\ 9y_1 \\ 9z_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7x_1 + 4y_1 - 4z_1 \\ 4x_1 - 8y_1 - z_1 \\ -4x_1 - y_1 - 8z_1 \end{bmatrix} = \begin{bmatrix} 9x_1 \\ 9y_1 \\ 9z_1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2x_1 + 4y_1 - 4z_1 \\ 4x_1 - 17y_1 - z_1 \\ -4x_1 - y_1 - 17z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{Reducing to row echelon form:}$$

$$\begin{bmatrix} 2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix} \xrightarrow{R_2 = R_2 + R_3} \begin{bmatrix} 2 & 4 & -4 \\ 4 & -17 & -1 \\ 0 & -18 & -18 \end{bmatrix}$$

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~~$$R_2 = R_2 - 2R_1 \rightarrow \begin{bmatrix} 2 & 4 & -4 \\ 0 & -9 & -9 \\ 0 & -9 & -9 \end{bmatrix}$$~~

~~$$\begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1, R_3 = R_3 + 2R_1} \begin{bmatrix} -2 & 4 & -4 \\ 0 & -25 & 7 \\ 0 & 7 & -25 \end{bmatrix}$$~~

~~$$\begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix} \xrightarrow{R_1 = R_2 + 2R_1, R_3 = R_3 - 2R_1} \begin{bmatrix} -2 & 4 & -4 \\ 0 & -9 & -9 \\ 0 & -9 & -9 \end{bmatrix}$$~~

$$R_3 = R_3 - R_2 \rightarrow \begin{bmatrix} -2 & 4 & -4 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

Now $\begin{bmatrix} -2 & 4 & -4 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow -9y_1 - 9z_1 = 0$
 $\Rightarrow y_1 = -z_1$

Let $z_1 = t \Rightarrow y_1 = -t$.

$$\Rightarrow -2x_1 + 4y_1 - 4z_1 = 0 \Rightarrow -2x_1 + 4(-t) - 4(t) = 0$$

$$\Rightarrow 2x_1 = -8t \Rightarrow x_1 = -4t$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} -4t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} \therefore \text{Eigenvector for } \lambda = 9 \text{ is } \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$$

Finding Eigenvector for $\lambda = -9$.

~~$$[A - \lambda I]x_2 = 0 \Rightarrow (A + 9I)x_2 = 0$$~~

$$\Rightarrow \begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Performing row reduction for $\begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix}$:

$$\begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 = 4R_2 - R_1 \\ R_3 = 4R_3 + R_1}} \begin{bmatrix} 16 & 4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16 & 4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow 4x_2 + y_2 - z_2 = 0.$$

$$\text{Let } z_2 = 4t_1 \text{ and } y_2 = 4t_2.$$

$$\Rightarrow x_2 = \frac{1}{4}(4t_1 - 4t_2) = t_1 - t_2.$$

$$\therefore \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} t_1 - t_2 \\ 4t_2 \\ 4t_1 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$\therefore$ Eigenvector for $\lambda = -9$ are

$$\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}.$$

The normalization for eigenvector gives

$$\begin{bmatrix} \frac{-4}{\sqrt{18}} \\ \frac{-1}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \\ 0 \end{bmatrix}$$

$\therefore$ The orthogonal matrix P is given by

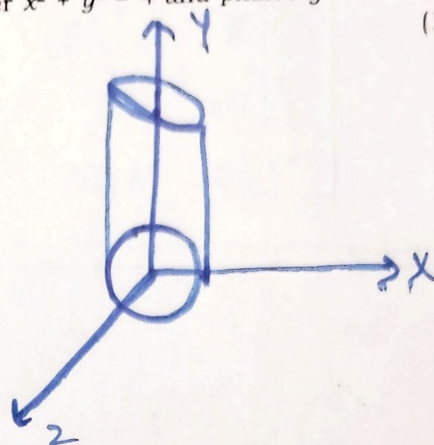
$$P = \begin{bmatrix} -4/\sqrt{17} & 1/\sqrt{17} & -1/\sqrt{17} \\ -1/\sqrt{17} & 0 & 4/\sqrt{17} \\ 1/\sqrt{17} & 4/\sqrt{17} & 0 \end{bmatrix} \quad \text{where } P^{-1}AP \text{ is diagonal matrix}$$



- Q.2 (b) Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and planes $y + z = 4$ and $z = 0$. (15)

Volume of the cylinder
is $V = \iiint dx dy dz$

Limit of $z = 0 \rightarrow 4 - y$



$$\Rightarrow V = \iint dx dy (4 - y) = \iint (4 - y) dx dy$$

Converting to polar coordinates,

$$x = r \cos \theta, \quad y = r \sin \theta, \quad dx dy = r dr d\theta$$

$$\Rightarrow V = \int_0^{2\pi} \int_0^2 (4 - r \sin \theta) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 (4r - r^2 \sin \theta) dr d\theta$$

$$= \int_0^{2\pi} \left(2r^2 - \frac{r^3 \sin \theta}{3} \right) d\theta = \int_0^{2\pi} \left(8 - \frac{8}{3} \sin \theta \right) d\theta$$

$$= \left(8\theta + \frac{8}{3} \cos \theta \right) \Big|_0^{2\pi} = \underline{\underline{16\pi}}$$

$\therefore$ Volume of cylinder = 16π

- Q.2 (c) A line with DR's (7, -5, 2) is drawn to intersect the lines $\frac{1}{-1}x(-7) = \frac{1}{1}(y+2) = \frac{1}{3}(z-5)$, $\frac{1}{2}(x-3) = \frac{1}{4}(y-5) = \frac{1}{-3}(z+3)$. Find the coordinates of the points of intersection and the length intercepted on it. (15)

Given lines : $\frac{7x}{1} = \frac{y+2}{1} = \frac{z-5}{3} \rightarrow L_1$

$\frac{x-3}{2} = \frac{y-5}{4} = \frac{z+3}{-3} \rightarrow L_2$

Any general point on L_1 is $\left(\frac{t}{7}, t-2, 3t+5\right)$

Any general point on L_2 is $(2s+3, 4s+5, -3s-3)$

The dir of line segment joining L_1 & L_2 is $\left(\frac{t}{7}-2s-3, t-4s-7, 3t+3s+8\right) \rightarrow (3)$

Line segment representing (3) is $\parallel$ to the line with dir (7, -5, 2):

$$\Rightarrow \frac{\frac{t}{7}-2s-3}{7} = \frac{t-4s-7}{-5} = \frac{3t+3s+8}{2}$$

Taking first 2 fractions,

$$-5(t-4s-7) = 49(t-2s-7)$$

$$\Rightarrow -5t+70s+105 = 49t-196s-343$$

$$54t-266s = 448$$

$$\Rightarrow 28t-133s = 224 \rightarrow (4)$$

Taking the last 2 fractions,

$$2(t-4s-7) = -5(3t+3s+8)$$

$$\Rightarrow 2t - 8s - 14 = -15t - 15s - 40$$

$$\Rightarrow 17t - 7s = -26 \quad \text{--- (3)}$$

~~Solving~~ Solving (4) & (3),

$$t = -\frac{718}{295}, \quad s = -\frac{648}{295}$$

Substituting these values in the general points of L_1 & L_2 , we get.

$$L_1 \Rightarrow \left(\frac{t}{7}, t-2, 3t+5 \right) = \left(\frac{-718}{2065}, \frac{-1308}{295}, \frac{-679}{295} \right)$$

$$L_2 \Rightarrow (2s+3, 4s+5, -3s-3) = \left(\frac{-411}{295}, \frac{-1117}{295}, \frac{1059}{295} \right)$$

Length intercepted = distance b/w the two points

$$\Rightarrow d = \sqrt{\left(\frac{-718}{2065} + \frac{411}{295} \right)^2 + \left(\frac{-1308}{295} + \frac{1117}{295} \right)^2 + \left(\frac{-679}{295} - \frac{1059}{295} \right)^2}$$

$$\Rightarrow d = \sqrt{\left(\frac{2799}{2065} \right)^2 + \left(\frac{1337}{2065} \right)^2 + \left(\frac{12166}{2065} \right)^2}$$

$$\Rightarrow \boxed{d \approx 6.08} \text{ units}$$

∴ The pts of intersection are :

$$\left(\frac{-78}{2065}, \frac{-1308}{295}, \frac{-679}{295} \right) \text{ and}$$

$$\left(\frac{-411}{295}, \frac{-1117}{295}, \frac{-1059}{295} \right) . \text{ The distance}$$

intercepted $\approx \underline{6.08}$ units



Q.3 (a)

When is a square matrix A said to be congruent to a square matrix B ? Prove that every matrix congruent to a skew symmetric is skew symmetric. (15)

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A square matrix A is similar to a square matrix B if $\exists$ matrix P where $|P| \neq 0$, and $B = P^T A P$.

Let A be a skew symmetric matrix and B be a congruent matrix to A

$$\Rightarrow A^T = -A$$

$$B = P^T A P. \quad \text{Taking Transpose on both sides, } B = (P^T A P)^T$$

$$\Rightarrow B^T = P^T A^T (P^T)^T = P^T (-A) P$$

$$\Rightarrow B^T = - (P^T A P) = -B$$

$$\boxed{\therefore B^T = -B}$$

$\Rightarrow B$ is also a skew symmetric matrix. Hence proved.

Q.3 (b)

The dimensions of the rectangular box are linear functions of time $l(t)$, $w(t)$ and $h(t)$. If the length and width are increasing at the rate 2 cm/sec and height is decreasing at the rate 3 cm/sec, find the rate at which the volume V and the surface area S are changing with respect to time. If $l(0) = 10$, $w(0) = 8$ and $h(0) = 20$, is V increasing or decreasing, when $t = 5$ sec? What about S when $t = 5$ sec? (15)

Volume of box is given by

$$V(t) = l(t) \cdot w(t) \cdot h(t). \text{ Differentiating both sides, } \frac{dV}{dt} = \left(\frac{dl}{dt}\right) \cdot w \cdot h + l \left(\frac{dw}{dt}\right) h + lw \left(\frac{dh}{dt}\right)$$

Given that $\frac{dl}{dt} = \frac{dw}{dt} = +2 \text{ cm s}^{-1}$ and

$$\frac{dh}{dt} = -3 \text{ cm s}^{-1} \quad \left(\begin{array}{l} l, w, h \text{ are the dimensions} \\ \text{at time } t \end{array} \right)$$

$$\Rightarrow \frac{dV}{dt} = 2wh + 2lh - 3lw$$

$$\therefore V \text{ is changing at rate of } \underline{2wh + 2lh - 3lw} \text{ cm}^3 \text{ s}^{-1}$$

Surface area of box is given by

$$S(t) = 2(lw + wh + lh)$$

$$\frac{dS}{dt} = 2 \left(\frac{dl}{dt} w + l \frac{dw}{dt} + w \frac{dh}{dt} + \frac{dw}{dt} h + l \frac{dh}{dt} + \frac{dl}{dt} h \right)$$

$$\Rightarrow \frac{dS}{dt} = 2(2w + 2l - 3w + 2h - 3l + 2h)$$

$$\frac{dS}{dt} = 2(4h - w - l)$$

$$\therefore S \text{ is changing at rate of } \underline{8h - 2w - 2l} \text{ cm}^2 \text{ s}^{-1}$$

Given that $l(0) = 10$, $w(0) = 8$, $h(0) = 20$.

$$\Rightarrow \frac{dl}{dt} = 2; \quad \frac{dw}{dt} = 2; \quad \frac{dh}{dt} = -3.$$

At $t = 5$,

$$l(5) = 10 + 5(2) = 20; \quad w(5) = 8 + 5(2) = 18;$$

$$h(5) = 20 + 5(-3) = 5.$$

$$\frac{dV}{dt} = 2wh + 2lh - 3lw. \quad \text{At } t = 5,$$

$$\begin{aligned} \frac{dV}{dt}(5) &= 2(18)(5) + 2(20)(5) - 3(20)(18) \\ &= 180 + 200 - 1080 = -700 \text{ cm}^3 \text{ s}^{-1} \end{aligned}$$

$$\therefore \frac{dV}{dt} = -700 \text{ cm}^3 \text{ s}^{-1} \Rightarrow V \text{ is decreasing at } t = 5 \text{ sec}$$

Now,

$$\frac{ds}{dt} = 8h - 2w - l \text{ cm}^2 \text{ s}^{-1}$$

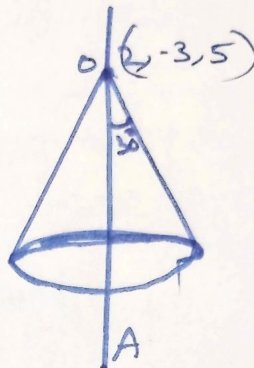
$$\begin{aligned} \Rightarrow \frac{ds}{dt} &= 8(5) - 2(18) - 20 = 40 - 36 - 20 \\ &= -16 \text{ cm}^2 \text{ s}^{-1}. \end{aligned}$$

$$\therefore \frac{ds}{dt} = -16 \text{ cm}^2 \text{ s}^{-1} \Rightarrow S \text{ is decreasing at } t = 5 \text{ sec}$$

- Q.3 (c) Find the equation to the right circular cone whose vertex is $(2, -3, 5)$, axis makes equal angles with the coordinate axes and semi vertical angle is 30° . (20)

Axis OA makes equal angles with the coord axes.

$\Rightarrow$ dir of OA are $(1, 1, 1)$



The eqn of cone with dir of axis (l, m, n) , Vertex (α, β, γ) and semi vertical angle θ is given by:

$$\left[l(x-\alpha) + m(y-\beta) + n(z-\gamma) \right]^2 = (l^2 + m^2 + n^2) \left[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 \right] (\tan^2 \theta)$$

$$\Rightarrow \left[1(x-2) + 1(y+3) + 1(z-5) \right]^2 = (1^2 + 1^2 + 1^2) \left[(x-2)^2 + (y+3)^2 + (z-5)^2 \right] \left(\frac{\sqrt{3}}{2} \right)^2$$

$$\Rightarrow (x+y+z-4)^2 = 3 \left[(x-2)^2 + (y+3)^2 + (z-5)^2 \right] \left(\frac{\sqrt{3}}{2} \right)^2$$

$$\Rightarrow 4(x+y+z-4)^2 = 9 \left[(x-2)^2 + (y+3)^2 + (z-5)^2 \right]$$

$$\Rightarrow 4(x^2 + y^2 + z^2 + 16 + 2xy + 2xz + 2yz - 8x - 8y - 8z)$$

$$= 9 \left[x^2 + y^2 + z^2 - 4x + 6y - 10z + 38 \right]$$

$$\Rightarrow 4x^2 + 4y^2 + 4z^2 + 8xy + 8yz + 8xz - 32x - 32y - 32z + 64 = 9x^2 + 9y^2 + 9z^2 - 36x + 54y - 90z + 342$$

$$\Rightarrow 5x^2 + 5y^2 + 5z^2 - 8xy - 8yz - 8xz - 4x + 86y - 58z + 278 = 0$$

is the reqd equation of cone.



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SECTION-B

- Q.5 (a) Given that $L\{f(t); p\} = F(p)$. Show that $\int_0^\infty \frac{f(t)}{t} dt = \int_0^\infty F(x) dx$. Hence evaluate the integral $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$. (10)

We know that $L\left(\frac{f(t)}{t}\right) = \int_p^\infty F(p) dp$

$$\Rightarrow \int_0^\infty e^{-pt} \frac{f(t)}{t} dt = \int_p^\infty F(p) dp.$$

$$\Rightarrow \int_0^\infty e^{-pt} \frac{f(t)}{t} dt = \int_p^0 F(p) dp + \int_0^\infty F(p) dp.$$

As $p \rightarrow 0$; $\int_p^0 F(p) dp = 0$

$$\Rightarrow \int_0^\infty \frac{f(t)}{t} dt = 0 + \int_0^\infty F(p) dp.$$

$$\Rightarrow \boxed{\int_0^\infty \frac{f(t)}{t} dt = \int_0^\infty F(p) dp.}$$

Calculating $L\left(\frac{e^{-t}}{t}\right)$.

$$L\left(\frac{e^{-t}}{t}\right) = \int_s^\infty L(e^{-t}) ds = \int_s^\infty \frac{1}{s+1} ds = \lim_{s \rightarrow \infty} \ln(s+1) - \ln(s+1)$$

$$L\left(\frac{e^{-3t}}{t}\right) = \int_s^\infty L(e^{-3t}) ds = \int_s^\infty \frac{1}{s+3} ds = \lim_{s \rightarrow \infty} \ln(s+3) - \ln(s+3)$$

$$\Rightarrow \cancel{L} L\left(\frac{e^{-t} - e^{-3t}}{t}\right) = -\ln(s+1) + \ln(s+3)$$

$$\lim_{s \rightarrow \infty} \ln\left(\frac{s+1}{s+3}\right)$$

$$\Rightarrow L\left(\frac{e^{-t} - e^{-3t}}{t}\right) = \ln\left(\frac{s+3}{s+1}\right) \quad \left(\because \lim_{s \rightarrow \infty} \ln\left(\frac{s+1}{s+3}\right) = \ln 1 = 0\right)$$

$$\text{Now } \int_0^{\infty} \frac{e^{-t} - e^{-3t}}{t} dt = \int_0^{\infty} \ln\left(\frac{s+3}{s+1}\right) ds$$

$$= \int_0^{\infty} \ln(s+3) - \ln(s+1) ds = \left[(s+3)\ln(s+3) - (s+3) \right. \\ \left. - (s+1)\ln(s+1) + (s+1) \right]_0^{\infty}$$

$$= \left[(s+3)\ln(s+3) - (s+1)\ln(s+1) - 2 \right]_0^{\infty}$$

Q.5 (b) Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3) dx - 3x^2y^2 dy$ along the path $x^4 - 6xy^3 = 4y^2$. (10)

Let the integral be written as

$$\int F \cdot dr \text{ where } F = (10x^4 - 2xy^3)\hat{i} + (-3x^2y^2)\hat{j}$$

Calculating $\nabla \times F$:

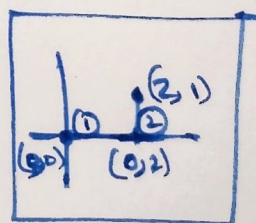
$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 10x^4 - 2xy^3 & -3x^2y^2 & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-6xy^2 - (-6xy^2)) = 0\hat{i} + 0\hat{j} + 0\hat{k}.$$

$\therefore \nabla \times F = 0$, the integral is independent of path.

Calculating along 2 paths:

$$(0,0) \rightarrow (2,0) \text{ and}$$

$$(2,0) \rightarrow (2,1)$$



① $(0,0) \rightarrow (2,0)$:

$$dy = 0 = y$$

$$\Rightarrow \int F \cdot dr = \int_0^2 10x^4 dx = \left(\frac{10x^5}{5} \right)_0^2 = \underline{\underline{64}}.$$

② $(2,0) \rightarrow (2,1)$

$$dx = 0; x = 2$$

$$\Rightarrow \int F \cdot dr = \int_0^1 -3 \times (2^2) y^2 dy$$

$$= -12 \int_0^1 y^2 dy = -12 \left(\frac{y^3}{3} \right)_0^1 = -4$$

$$\therefore \int_{(0,0)}^{(2,1)} F \cdot dr = 64 - 4 = 60 \text{ along the Path.}$$

$$\therefore \int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3) dx - 3x^2 y^2 dy = 60$$

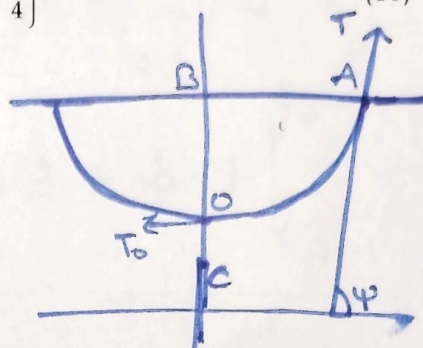
- Q.5 (c) A uniform chain of length l , which can just bear a tension of n times its weight, is stretched between two points on the same horizontal line. Show that the least possible sag in the middle is $l \left\{ n - \sqrt{n^2 - \frac{1}{4}} \right\}$. (10)

$$T_A = wnl, \quad T_A = wy$$

$$T_0 = wc$$

$$\text{Also, } y_A = nl \quad \text{--- (1)}$$

$$T \cos \psi = T_0 = wc \quad \text{--- (2)}$$



(w = weight per unit length)

$$\text{At A, } S = l/2$$

$$S = c \tan \psi \Rightarrow \frac{l}{2} = c \tan \psi$$

$$\Rightarrow l = 2c \tan \psi \quad \text{--- (3)}$$

$$\text{In (2), } T \cos \psi = wc = wnl \cos \psi$$

$$\Rightarrow c \sec \psi = nl; \quad c \tan \psi = \frac{l}{2}$$

$$\Rightarrow c^2 = n^2 l^2 - \frac{l^2}{4} \Rightarrow c = l \sqrt{n^2 - \frac{1}{4}}$$

The sag in the middle is given by:

$$\text{Sag} = y_A - c = nl - l \sqrt{n^2 - \frac{1}{4}}$$

$$\Rightarrow \text{sag} = l \left(n - \sqrt{n^2 - \frac{1}{4}} \right)$$

Hence proved

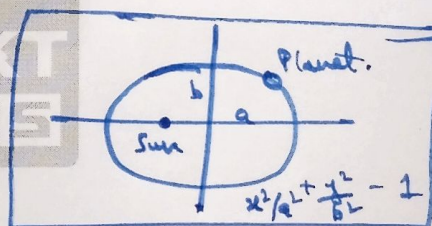
Q.5 (d) State Kepler's laws of Planetary Motion.

(10)

Candidates must not write on this margin

Kepler's laws of Planetary motion.(I) Kepler's 1st law

All planets revolve around the sun in an elliptical path where the sun is situated at one of the focal point

(II) Kepler's second law

The planets revolving around the sun sweep equal areas in equal intervals of time.

$$\Rightarrow \frac{dA}{dt} = \text{constant}$$

III) Kepler's third law

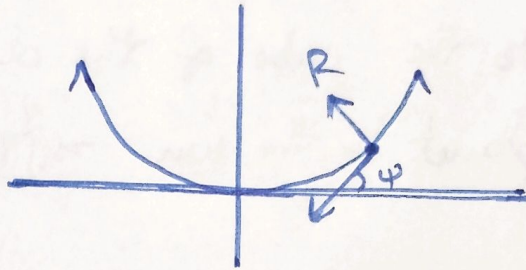
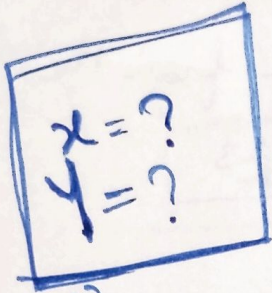
The ^{square of the} time period of revolution of a planet around the sun is directly proportional to the cube of the distance from the planet to the sun $\Rightarrow \boxed{T^2 \propto R^3}$



Q.5 (e)

If a particle slides down a smooth cycloid, starting from a point whose actual distance from the vertex is b . Prove that its speed at any time t is $\frac{2xb}{Y} \sin\left(\frac{2xt}{T}\right)$, where T is the time of complete oscillation of the particle. (10)

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$$m \frac{d^2 s}{dt^2} = -mg \sin \phi$$

$$\Rightarrow \frac{d^2 s}{dt^2} = -g \sin \phi$$

Multiplying both sides with $2 \frac{ds}{dt}$ and integrating,

$$2 \frac{ds}{dt} \frac{d^2 s}{dt^2} = -2g \frac{ds}{dt} \sin \phi$$

$$\Rightarrow \frac{d}{dt} \left[\left(\frac{ds}{dt} \right)^2 \right] = -2g \frac{ds}{dt} \sin \phi$$

$$\Rightarrow \left(\frac{ds}{dt} \right)^2 = -2g \int ds \sin \phi$$

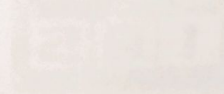
~~$\left(\frac{ds}{dt} \right)^2 = -2g s \sin \phi$~~ we know that $s = 4a \sin \phi$

$$\Rightarrow \left(\frac{ds}{dt} \right)^2 = \frac{-2g}{4a} \times \frac{s^2}{2} = \frac{-g s^2}{4a} + A$$

$$\Rightarrow V^2 = \left(\frac{ds}{dt}\right)^2 = \frac{-gs^2}{4a} + A. \quad \text{At } s=b, V=0$$

$$\Rightarrow 0 = -\frac{gb^2}{4a} + A \Rightarrow A = \frac{gb^2}{4a}$$

$$\Rightarrow V^2 = \frac{g}{4a} (b^2 - s^2) \quad \Rightarrow V = \sqrt{\frac{g}{4a} (b^2 - s^2)}$$



Q.7 (a) Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$;

$a > b > 0$ are constants, and λ is a parameter. Show that the given family of curves is self-orthogonal.

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{--- (1)} \quad (15)$$

Differentiating the eqn w.r.t x ,

$$\frac{2x}{a^2+\lambda} + \frac{2yy'}{b^2+\lambda} = 0$$

$$\Rightarrow \frac{x}{a^2+\lambda} = -\frac{yy'}{b^2+\lambda} \Rightarrow xb^2+x\lambda = -a^2yy' - \lambda yy'$$

$$\Rightarrow \lambda(x+yy') = -xb^2 - a^2yy'$$

$$\Rightarrow \lambda = \frac{-xb^2 - a^2yy'}{x+yy'} \quad \text{Substituting}$$

the value of λ in (1), we get the D.E of the curves.

$$\begin{aligned} a^2+\lambda &= a^2 - \frac{xb^2 + a^2yy'}{x+yy'} \\ &= \frac{x(a^2-b^2)}{(x+yy')} \end{aligned}$$

$$\begin{aligned} b^2+\lambda &= b^2 - \frac{xb^2 + a^2yy'}{x+yy'} \\ &= \frac{(b^2-a^2)yy'}{x+yy'} \end{aligned}$$

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

$$\Rightarrow \frac{x(x+yy')}{a^2-b^2} + \frac{y(x+yy')}{y'(b^2-a^2)} = 1$$

$$\Rightarrow x(x+yy') - \frac{y}{y'}(x+yy') = a^2-b^2$$

$$\Rightarrow \boxed{(x+yy')(x - \frac{y}{y'}) = a^2-b^2} \text{ is the reqd DE of curve.} \quad \hookrightarrow \textcircled{2}$$

For orthogonal trajectories, replacing y' with $-\frac{1}{y'}$, we get:

$$(x - \frac{y}{y'}) (x + yy') = a^2 - b^2 \quad \text{---} \textcircled{3}$$

$$\textcircled{3} = \textcircled{2}$$

Thus the curve & its orthogonal trajectory have same DE.

$\Rightarrow$ Curve is self orthogonal. Hence proved.

Eqn of orthogonal trajectory:

$$\boxed{\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} = 1}$$

Q.7 (b) If $\vec{F} = \left(y \frac{\partial \phi}{\partial z} - z \frac{\partial \phi}{\partial y}\right) \hat{i} + \left(z \frac{\partial \phi}{\partial x} - x \frac{\partial \phi}{\partial z}\right) \hat{j} + \left(x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x}\right) \hat{k}$, then prove that

$$\vec{F} - (\vec{r} \times \nabla \phi) = \vec{F}, \vec{r} = \vec{F}, \nabla \phi = 0. \quad (15)$$

Calculating $\vec{F} - (\vec{r} \times \nabla \phi)$:

$$\vec{r} \times \nabla \phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \hat{i} \left(y \frac{\partial \phi}{\partial z} - z \frac{\partial \phi}{\partial y}\right) + \hat{j} \left(z \frac{\partial \phi}{\partial x} - x \frac{\partial \phi}{\partial z}\right) + \hat{k} \left(x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x}\right)$$

$$\vec{F} - (\vec{r} \times \nabla \phi) = \hat{i}(0) + \hat{j}(0) + \hat{k}(0) = 0.$$

$$\therefore \vec{F} - (\vec{r} \times \nabla \phi) = 0 \quad \text{--- (1)}$$

Calculating $\vec{F} \cdot \vec{r}$:

$\Rightarrow$ From (1), $\vec{r} \times \nabla \phi = \vec{F}$.

$$\vec{F} \cdot \vec{r} = \vec{r} \cdot (\vec{r} \times \nabla \phi) = 0$$

$$\boxed{\therefore \vec{F} \cdot \vec{r} = 0} \quad \text{--- (2)} \quad \left(\because \vec{A} \cdot (\vec{A} \times \vec{B}) = 0 \right) \rightarrow \text{Triple product} = 0 \quad [AAB]$$

Calculating $\vec{F} \cdot \nabla \phi$:

$$\vec{F} \cdot \nabla \phi = (\vec{r} \times \nabla \phi) \cdot \nabla \phi = 0$$

$$\boxed{\therefore (\vec{A} \times \vec{B}) \cdot \vec{B} = 0} \rightarrow \text{Triple product} [ABB] = 0$$

$$\boxed{\therefore F \cdot \nabla \phi = 0} - (3)$$

From (1), (2), (3)

$$\boxed{F - (a \times \nabla \phi) = F \cdot a = F \cdot \nabla \phi = 0}$$

hence proved.

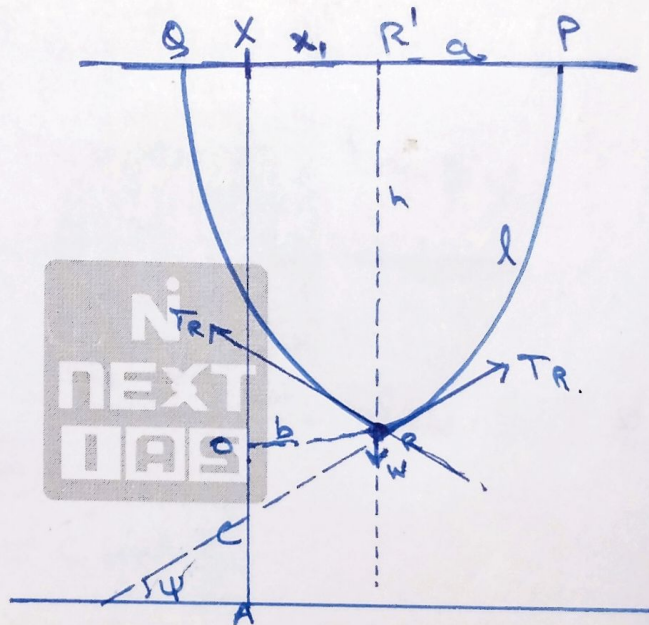


Q.7 (c)

PR and QR are two heavy strings tied together at R and carrying a weight W at R . P and Q are two points in the same horizontal line and $2a$ is the distance between them. l is the length of each string and h is the depth of R below PQ . Prove that

$$(i) \quad l^2 - h^2 = 2c^2 \left(\cosh \frac{a}{c} - 1 \right)$$

$$(ii) \quad \text{Tension at } P \text{ or } Q = \frac{1}{2h} \{ lw + (l^2 + h^2)w \}, \text{ where } c, \text{ is the parameter of the catenary and } w \text{ is the line density of the string.} \quad (20)$$



$$\text{Let } XR' = x_1 \\ R'P = a \text{ (given)}$$

Extending the string to form an imaginary catenary,

$$2T_R \sin \psi = W \Rightarrow T_R = \frac{W}{2} \csc \psi$$

$$\text{Let } OR = b \text{ and } RP = l \text{ (given)}$$

$$T_R = w y_R = \frac{W}{2} \csc \psi$$

$$\Rightarrow y_R = \frac{W}{2w} \csc \psi$$

$$\text{Also, } T_R \cos \psi = T_0$$

$$\Rightarrow \frac{W}{2} \frac{1}{\sin \psi} \times \cos \psi = T_0$$

$$\Rightarrow T_0 = \frac{W}{2} \cot \psi = wc$$

$$\Rightarrow c = \frac{W}{2wc} \cot \psi$$

From the catenary, $S = c \tan \psi$

$$\Rightarrow \text{At R, } S = b = c \times \tan \psi$$

$$\Rightarrow b = \frac{W}{2wc} \cot \psi \times \tan \psi \Rightarrow b = \frac{W}{2wc}$$

$$\Rightarrow \boxed{b = \frac{W}{2wc}} \text{ Finding the values of } l \text{ and } h$$

$$\text{We know } S = c \sinh \frac{x}{c}$$

$$\text{At R, } S = c \sinh \left(\frac{x_1}{c} \right) \text{ ; At P, } S = c \sinh \left(\frac{x_1 + a}{c} \right)$$

$$\Rightarrow b = c \sinh \frac{x_1}{c} \text{ ; } \Rightarrow l + b = c \sinh \left(\frac{x_1 + a}{c} \right)$$

$$\Rightarrow l = c \left[\sinh \left(\frac{x_1 + a}{c} \right) - \sinh \left(\frac{x_1}{c} \right) \right] \text{ --- (1)}$$

$$\text{We know } y = c \cosh \frac{x}{c}$$

$$\text{At R, } y = c \cosh \left(\frac{x_1}{c} \right) \text{ ; At P, } y = c \cosh \left(\frac{x_1}{c} \right)$$

$$\Rightarrow y_2 = c \cosh \frac{x_1}{c} \text{ ; } \Rightarrow y_2 + h = c \cosh \left(\frac{x_1 + a}{c} \right)$$

$$\Rightarrow h = c \left[\cosh \left(\frac{x_1 + a}{c} \right) - \cosh \left(\frac{x_1}{c} \right) \right] \text{ --- (2)}$$

From (1) & (2), Calculating $l^2 - h^2$,

$$l^2 - h^2 = c^2 \left[\left[\sinh\left(\frac{x_1+a}{c}\right) - \sinh\left(\frac{x_1}{c}\right) \right]^2 - \left[\cosh\left(\frac{x_1+a}{c}\right) - \cosh\left(\frac{x_1}{c}\right) \right]^2 \right]$$

$$\Rightarrow \cancel{l^2 - h^2} = c^2 \left[-1 - 1 + 2 \cosh\left(\frac{x_1+a}{c}\right) \cosh\left(\frac{x_1}{c}\right) - 2 \sinh\left(\frac{x_1+a}{c}\right) \sinh\left(\frac{x_1}{c}\right) \right]$$

$$\Rightarrow l^2 - h^2 = c^2 \left[-2 + 2 \cosh\left(\frac{x_1+a}{c} - \frac{x_1}{c}\right) \right]$$

~~As~~

$$\left(\because \cosh^2 x - \sinh^2 x = 1 \text{ and } \cosh a \cosh b - \sinh a \sinh b = \cosh(a-b) \right)$$

$$\Rightarrow l^2 - h^2 = c^2 \left[2 \cosh\left(\frac{a}{c}\right) - 2 \right]$$

$$\therefore l^2 - h^2 = 2c^2 \left[\cosh\left(\frac{a}{c}\right) - 1 \right]$$

Tension at P is $= w y_p$

We know that $y_p^2 = c^2 + p^2$, $y_R^2 = c^2 + s^2$

$$\Rightarrow (y_R + h)^2 = c^2 + (b + l)^2; \quad y_R^2 = c^2 + b^2$$

Subtracting, we get;

$$y_R^2 + h^2 + 2y_R h - y_R^2 = c^2 + b^2 + l^2 + 2bl - c^2 - b^2$$

$$\Rightarrow h^2 + 2y_R h = l^2 + 2bl. \text{ Now, } 2b = \frac{W}{w} \quad (\text{From (1)})$$

$$\Rightarrow h^2 + 2y_R h = l^2 + \frac{Wl}{w}$$

$$\Rightarrow 2y_R h = (l^2 - h^2) + \frac{Wl}{w} \Rightarrow y_R = \frac{l^2 - h^2}{2h} + \frac{Wl}{2wh}$$

$$\text{Now } y_p = y_R + h$$

$$\Rightarrow y_p = \frac{l^2 - h^2}{2h} + h + \frac{Wl}{2wh} = \frac{l^2 + h^2}{2h} + \frac{Wl}{2wh}$$

~~y_p~~ Tension at $y_p = w y_p$

$$\Rightarrow T_p = w \left(\frac{l^2 + h^2}{2h} + \frac{Wl}{2wh} \right)$$

$$T_p = \left(\frac{1}{2h} \right) \left((l^2 + h^2)w + Wl \right)$$

Hence proved -