

NEXT IAS

(To be filled by candidate)

Name of Candidate : ANNU KOUSIK NARASIMHA
 Roll No. : OAC24MTTA003 Contact No. [REDACTED]
 e-mail: [REDACTED] GMAIL.COM Date of Examination : 02/08/2024
 Exam Centre : Old Rajinder Nagar ☐ Online ☒

CSE (MAINS) TEST SERIES-2024

Test-06	MATHEMATICS
Dated : 28-07-2024	OPTIONAL

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS*(Please read each of the following instructions carefully before attempting questions)**There are EIGHT questions divided in Two sections and printed in ENGLISH.**Candidate has to attempt FIVE questions in all.**Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.**The number of marks carried by a question/part is indicated against it.**Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.**No marks will be given for answers written in a medium other than the authorized one.**Word limit in questions, wherever specified, should be adhered to.**Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.***DELHI CENTRE:****Old Rajinder Nagar:** 27-B, Pusa Road, Metro Pillar no. 118, Near Karol Bagh Metro, New Delhi-110060 | Phone: 8081300200**Mukherjee Nagar:** 1422, Main Mukherjee Nagar Road, Near Batra Cinema, New Delhi-110009 | Phone: 8081300200**BHOPAL CENTRE:** Plot No. 46 Zone - 2, M.P Nagar, Bhopal - 462011 | Phone: 8827664612, 8081300200**JAIPUR CENTRE:** Plot No. 6 & 7, 3rd Floor, Sree Gopal Nagar, Gopalpura Bypass, Jaipur - 302015 | Phone: 9358200511**PRAYAGRAJ CENTRE:** 31/31 Sardar Patel Marg, Civil Lines, Prayagraj, Uttar Pradesh - 211001 | Phone: 9958857757

(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	12			
2. (c)	19	13			
2. (d)	21	10			
3. (a)	23	15			
3. (b)	26	15			
3. (c)	29	20			
4. (a)	33	15			
4. (b)	36	15			
4. (c)	39	20			
5. (a)	43	10			
5. (b)	45	10			
5. (c)	47	10			
5. (d)	49	10			
5. (e)	51	10			
6. (a)	53	18			
6. (b)	57	15			
6. (c)	60	17			
7. (a)	64	20			
7. (b)	68	15			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	80	20			

Remarks

Observations:

SECTION-A

- Q.1 (a) Let R be a commutative ring with unity such that $a^2 = a, \forall a \in R$. If I be any prime ideal of R , find all the elements of R/I . (10)

Given: R is boolean ring

I is prime Ideal of R .

We know that if P is prime ideal of Ring R , then $\frac{R}{P}$ is an integral domain.

$\therefore \frac{R}{I}$ is an integral domain.

Let K be an integral domain such that $a^2 = a \forall a \in K$.

The elements of K (idempotent elements):

$$a^2 = a \Rightarrow a \cdot a = a \cdot 1 \Rightarrow a = 1$$

(left cancellation law)

Also $a = 0$ clearly satisfies the condition

$\therefore$ The only elements are $a = 0, a = 1$

$\therefore$ The elements of $\frac{R}{I}$ are

$$\{0+I, 1+I\}$$

Candidates
must not write
on this margin



Q.1 (b)

Determine for each of the sets $\left\{1, 1 + \frac{(-1)^n}{10^n} : n \in \mathbb{N}\right\}$, $\left\{1 - \frac{2}{n} : n \in \mathbb{N}\right\}$, $\left\{\frac{m}{n} : m, n \in \mathbb{N}\right\}$

I. The suprema and infima

II. The limit points

III. The lower and upper limits. Specify which of the above sets are closed or open? (10)

$$\textcircled{I} \text{ set } \left\{1, 1 + \frac{(-1)^n}{(10)^n} : n \in \mathbb{N}\right\}$$

Elements of the set: $\left\{1, 1 - \frac{1}{10}, 1 + \frac{1}{100}, 1 - \frac{1}{1000}, \dots\right\}$

~~Other~~ elements of set $\Rightarrow \left\{1, 1 + \frac{1}{(10)^{2n}} : n \in \mathbb{N}\right\}$

$$\therefore \text{Suprema} = 1 + \frac{1}{100} = \frac{101}{100}$$

~~Other~~ elements of set $= \left\{1, 1 - \frac{1}{(10)^{2n+1}} : n \in \mathbb{N}\right\}$

$$\text{Infima} = 1 - \frac{1}{10} = \frac{9}{10}$$

Limit point: The ~~are~~ elements less than and greater than 1 convg to: $\lim_{n \rightarrow \infty} 1 + \frac{(-1)^n}{10^n} = \underline{1}$

$$\therefore \text{Limit pt} = \underline{1}$$

$$\text{Here, } \underline{\text{Lower Limit}} = \underline{\text{Upper Limit}} = \underline{1}$$

The set is closed set since the lower & upper bound are included in the set.

$$\textcircled{II} \text{ set } \left\{1 - \frac{2}{n} : n \in \mathbb{N}\right\}$$

Elements of set $\Rightarrow \left\{-1, 0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \dots\right\}$

$$\text{Suprema at } n \rightarrow \infty = \underline{1} \quad (\text{decreasing sequence})$$

$$\text{Infima at } n=1 = \underline{-1}$$

Limit points: as $n \rightarrow \infty$: $\text{Limit} = \lim_{n \rightarrow \infty} 1 - \frac{2}{n} = 1$

$\Rightarrow$ Limit point = 1

~~Lower limit~~ Upper limit = 1. Lower limit does not exist.

The set is closed set, as the suprema and infima occur in the set itself.

III) set $\left\{ \frac{m}{n} : m, n \in \mathbb{N} \right\}$

Suprema at ∞ ; Infima at 0

The set is non convergent, hence limit points do not exist.

The lower & upper limits do not exist.

The set is an open set, as the bounds are not included in this set.

Q.1 (c) Show that $1 - \frac{1}{4 \cdot 3} + \frac{1}{4^2 \cdot 5} - \frac{1}{4^3 \cdot 7} + \dots$ is convergent.

(10)

Let $S_n = \frac{(-1)^{n+1}}{4^{n-1}(2n-1)}$ be the element of the ~~series~~ series.

Checking for absolute convergence of the series.

$$\text{Let } U_n = \frac{1}{4^{n-1}(2n-1)} \Rightarrow U_{n+1} = \frac{1}{4^n(2n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{U_n}{U_{n+1}} = \lim_{n \rightarrow \infty} \frac{1/4^{n-1}(2n-1)}{1/4^n(2n+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{4^n(2n+1)}{4^{n-1}(2n-1)} = \lim_{n \rightarrow \infty} 4 \frac{(2 + 1/n)}{(2 - 1/n)}$$

$$= 4 > 1$$

$$\therefore \lim_{n \rightarrow \infty} \frac{U_n}{U_{n+1}} > 1$$

Hence By ratio test the series $|S_n|$ is convergent.

$$\Rightarrow \sum_{n=1}^{\infty} |S_n| \text{ is convg} \Rightarrow \sum_{n=1}^{\infty} S_n \text{ is } \underline{\underline{\text{absolutely convergent}}}$$

We know that absolute convergence
implies regular convergence

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^{n-1}(2n-1)} \text{ is convergent.}$$

Hence proved.



Q.1 (d) If $v(r, \theta) = \left(r - \frac{1}{r}\right) \sin \theta$, $r \neq 0$, then find analytic function $f(z) = u(r, \theta) + iv(r, \theta)$. (10)

Finding $u(r, \theta)$ by CR equations.

We know
$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r} \left(r - \frac{1}{r}\right) \cos \theta = \left(1 - \frac{1}{r^2}\right) \cos \theta$$

$$\Rightarrow du = \left(1 - \frac{1}{r^2}\right) \cos \theta dr$$

$$\Rightarrow u = \left(r + \frac{1}{r}\right) \cos \theta + f(\theta) \quad \text{--- (1)}$$

Also
$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

$$\frac{\partial v}{\partial r} = -\left(1 + \frac{1}{r^2}\right) \sin \theta; \quad \frac{\partial u}{\partial \theta} = -r \left(1 + \frac{1}{r^2}\right) \sin \theta$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = \left(-r - \frac{1}{r}\right) \sin \theta \quad \text{--- (2)}$$

$$\text{From (1), } \frac{\partial u}{\partial \theta} = \left(-r - \frac{1}{r}\right) \sin \theta + f'(\theta) \quad \text{--- (3)}$$

Comparing (2) & (3), $f'(\theta) = 0$

$$\Rightarrow f(\theta) = C$$

$$\Rightarrow u(r, \theta) = \left(r + \frac{1}{r}\right) \cos \theta + C$$

$$f(z) = u(r, \theta) + i v(r, \theta)$$

$$\Rightarrow f(z) = \left[\left(r + \frac{1}{r} \right) \cos \theta + C \right] + i \left[\left(r - \frac{1}{r} \right) \sin \theta \right]$$

$$\Rightarrow f(z) = \left(r \cos \theta + i r \sin \theta \right) + \left(\frac{\cos \theta}{r} - \frac{i \sin \theta}{r} \right) + C$$

$$\Rightarrow f(z) = r (\cos \theta + i \sin \theta) + \frac{1}{r} (\cos \theta - i \sin \theta) + C$$

$$\Rightarrow f(z) = r e^{i\theta} + \frac{1}{r} e^{-i\theta} + C$$

$$\Rightarrow f(z) = z + \frac{1}{z} + C \quad (\because z = r e^{i\theta})$$

$\therefore$ the reqd analytic function.

Q.1 (e)

Using graphical method, solve the linear programming problem.

$$\text{Maximise } Z = 3x_1 + 2x_2$$

$$\text{Subject to } x_1 - x_2 \geq 1,$$

$$x_1 + x_3 \geq 3$$

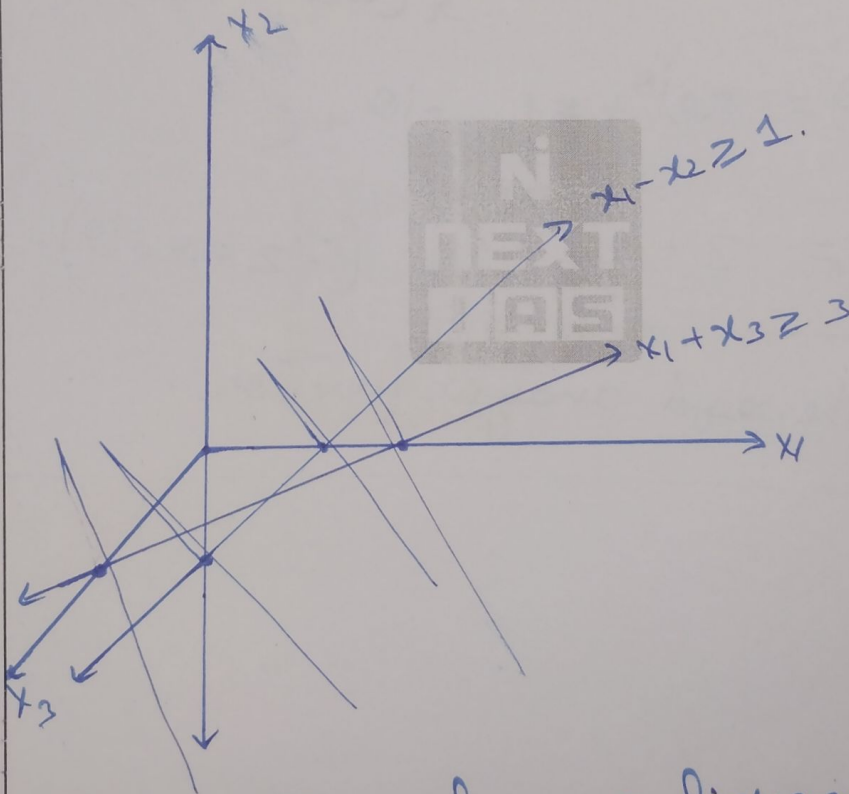
$$x_1, x_2, x_3 \geq 0$$

(10)

$$\text{Max } Z = 3x_1 + 2x_2; \text{ Conditions: } x_1 - x_2 \geq 1$$

$$x_1 + x_3 \geq 3$$

Drawing the planes in 3D axes:



The problem shows a divergent solution as $x_1 \rightarrow \infty$ and $x_2 = 0, x_3 \rightarrow \infty$ satisfying the conditions $x_1 - x_2 \geq 1$ & $x_1 + x_3 \geq 3$

$$\therefore \text{Max } Z = 3x_1 + 2x_2 \rightarrow \infty$$

Candidates must not write on this margin

Candidates
must not write
on this margin



Q.2 (a)

Obtain the maxima and minima of $x^2 + y^2 + z^2 - yz - zx - xy$ subject to the condition

$$x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 = 0. \quad (15)$$

Let $f(x, y, z) = x^2 + y^2 + z^2 - yz - zx - xy$ and

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 = 0.$$

By Lagrange's method of multipliers,

$$d(f + \lambda \phi) = 0 \Rightarrow \frac{\partial}{\partial x}(f + \lambda \phi) dx + \frac{\partial}{\partial y}(f + \lambda \phi) dy + \frac{\partial}{\partial z}(f + \lambda \phi) dz = 0.$$

$$\Rightarrow \frac{\partial}{\partial x}(f + \lambda \phi) = 0, \frac{\partial}{\partial y}(f + \lambda \phi) = 0, \frac{\partial}{\partial z}(f + \lambda \phi) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow (2x - z - y) + \lambda(2x - 2) = 0.$$

$$\Rightarrow 2x(1 + \lambda) - y - z = 2\lambda \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow (2y - z - x) + \lambda(2y + 2) = 0$$

$$\Rightarrow 2y - x + 2y(1 + \lambda) - z = -2\lambda \quad \text{--- (2)}$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow (2z - y - x) + \lambda(2z + 6) = 0$$

$$\Rightarrow -x - y + 2z(1 + \lambda) = -6\lambda \quad \text{--- (3)}$$

$$\textcircled{1} - \textcircled{2} \Rightarrow (3x + 2z\lambda) - 3y - 2y\lambda = 4$$

$$\Rightarrow x(3 + 2\lambda) + y(-3 - 2\lambda) = 4$$

$$\Rightarrow (x - y) = \frac{4\lambda}{3 + 2\lambda} \quad \text{--- (4)}$$

$$\textcircled{2} - \textcircled{5} \Rightarrow 3y + 2y\lambda - 3z - 2z\lambda = 4$$

$$\Rightarrow y(3+2\lambda) + 2(-z-2\lambda) = 4$$

$$\Rightarrow y - z = \frac{4\lambda}{3+2\lambda} - \textcircled{5}$$

$$\textcircled{1} - \textcircled{5} \Rightarrow 3x + 2x\lambda - 3z - 2z\lambda = 8$$

$$\Rightarrow x(3+2\lambda) - z(3+2\lambda) = 8$$

$$\Rightarrow x - z = \frac{8\lambda}{3+2\lambda} - \textcircled{6}$$

To find: $x^2 + y^2 + z^2 - xy - yz - xz$

$$= x(x-y) + y(y-z) + z(z-x)$$

$$= x\left(\frac{4\lambda}{3+2\lambda}\right) + y\left(\frac{4\lambda}{3+2\lambda}\right) + z\left(\frac{-8\lambda}{3+2\lambda}\right)$$

$$= (x+y-2z)\left(\frac{4\lambda}{3+2\lambda}\right)$$

$$\textcircled{1} \Rightarrow 2x\lambda + (x-y) + z = 2\lambda$$

$$\Rightarrow 2x\lambda + \left(\frac{4\lambda}{3+2\lambda}\right) + \left(\frac{8\lambda}{3+2\lambda}\right) = 2\lambda$$

$$\Rightarrow 2x\lambda = 2 - \frac{12\lambda}{3+2\lambda} \Rightarrow x\lambda = 1 - \frac{6\lambda}{3+2\lambda}$$

$$\Rightarrow x\lambda = \frac{(2\lambda-3)\lambda}{3+2\lambda} \Rightarrow \boxed{x = \frac{2\lambda-3}{3+2\lambda}}$$

$$\Rightarrow y = x - \frac{4\lambda}{3+2\lambda} = \underline{\underline{-1}}$$

$$\text{also } z = x - \frac{8\lambda}{3+2\lambda} = \frac{-6\lambda-3}{(3+2\lambda)} = z$$

$$\text{Now } x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 = 0.$$

$$\Rightarrow \frac{(2\lambda-3)^2}{(2\lambda+3)^2} + \frac{1}{\lambda^2} + \frac{(6\lambda+3)^2}{\lambda^2(2\lambda+3)^2} + \frac{6-4\lambda}{\lambda(3+2\lambda)} - \frac{2}{\lambda} + \frac{-36\lambda-18}{\lambda(3+2\lambda)} + 9 = 0.$$

$$\Rightarrow (2\lambda-3)^2 + (2\lambda+3)^2 + (6\lambda+3)^2 + (6-4\lambda)(2\lambda+3\lambda) - 2\lambda(2\lambda+3)^2 + (-36\lambda-18)(2\lambda^2+3\lambda) + 9\lambda^2(2\lambda+3)^2 = 0$$

$$\Rightarrow 8\lambda^2 + 18 + 36\lambda^2 + 9 + 36\lambda + 18\lambda - 8\lambda^3 - 8\lambda^3 - 18\lambda - 24\lambda^2 + (-72\lambda^3 - 144\lambda^2 - 54\lambda + 36\lambda^4 + 81\lambda^2 - 108\lambda^3) = 0$$

$$\Rightarrow \frac{(2\lambda-3)^2}{(2\lambda+3)^2} + 1 + \frac{(6\lambda+3)^2}{(2\lambda+3)^2} + \frac{-4\lambda+6}{(2\lambda+3)} - \frac{2}{\lambda} + \frac{(-36\lambda-18)}{(3+2\lambda)} + 9 = 0.$$

$$\Rightarrow (2\lambda-3)^2 + (2\lambda+3)^2 + (6\lambda+3)^2 + (-4\lambda+6)(2\lambda+3) + (-36\lambda-18)(2\lambda+3) + 9(2\lambda+3)^2 = 0.$$

$$\Rightarrow \text{Solving, we get } \lambda = 3/2, -9/2$$

$$\lambda = 3/2 \Rightarrow x = 0, y = -1, z = -2$$

$$\Rightarrow f = 3 \Rightarrow \boxed{\text{Min } f = 3} \text{ at } (0, -1, -2)$$

$$\lambda = -9/2 \Rightarrow x = 2, y = -1, z = -4$$

$$\Rightarrow f = 27 \Rightarrow \boxed{\text{Max } f = 27} \text{ at } (2, -1, -4)$$

Q.2 (b)

Show that the set of all polynomials with even coefficients is a prime ideal in $\mathbb{Z}[x]$. (12)Candidates
must not write
on this margin

$$\text{Let } P = \{a_0 + a_1x + a_2x^2 + \dots + a_nx^n\}$$

P is a prime ideal if $\frac{\mathbb{Z}[x]}{P}$ is an integral domain.

$\frac{\mathbb{Z}[x]}{P}$ is a commutative ring with unity

To prove: $\frac{\mathbb{Z}[x]}{P}$ has no zero divisors

$$\text{Elements of } \frac{\mathbb{Z}[x]}{P} = \{Z + P_1, Z + P_2, \dots\}$$

$$\text{Let } Z + P_1, Z + P_2 \in \frac{\mathbb{Z}[x]}{P}$$

$$(Z + P_1)(Z + P_2) = 0 + Z \text{ then } (Z + P_1)(Z + P_2) = Z$$

$$\Rightarrow Z + Z(P_1 + P_2) + P_1P_2 = 0 + Z$$

$$\Rightarrow \cancel{P_1P_2} + Z + P_1P_2 = 0 + Z$$

$$\Rightarrow P_1P_2 = 0 \in \mathbb{Z}$$

Candidates
must not write
on this margin



Q.2 (c)

Let I and J be ideals in a ring R . Then prove that the quotient ring $(I+J)/J$ is isomorphic to the quotient ring $\frac{I}{I \cap J}$. (13)

Candidates must not write on this margin

Since I, J are ideals, $I+J$ is a subring and J is an ideal of $\frac{I+J}{J}$.

Consider a function $\phi: \frac{I+J}{J} \rightarrow I$.

Elements of $\frac{I+J}{J} = \left\{ \frac{i_1+j_1}{J}, \frac{i_2+j_2}{J}, \dots \right\}$

are given by $\Rightarrow \{(i_1+j_1)+J, (i_2+j_2)+J, \dots\}$

where $i_k, j_k \in I, J$ respectively.

But $j_k \in J \Rightarrow j_k + J = J$.

$\therefore$ Elements of $\frac{I+J}{J}$ are $\{i_1+J, i_2+J, \dots\}$

Consider the function ϕ as $\boxed{\phi(i_k+J) = i_k}$

Let $i_1, i_2 \in I$.

Then $\phi(i_1+J + i_2+J) = \phi(i_1+i_2+J) = i_1+i_2$

$= \phi(i_1+J) + \phi(i_2+J)$

$\therefore \underline{\phi(a+b) = \phi(a) + \phi(b)} \Rightarrow \phi$ is a homomorphism.

Also let $i_k \in I$. Then $\phi(i_k+J) = i_k$

$\therefore \forall i_k \in I, \exists i_k+J \in \frac{I+J}{J} \Rightarrow \phi(i_k+J) = i_k$

$\therefore \phi$ is onto.

By Fundamental Theorem of ~~the~~ isomorphism,
Every homomorphic image of R is isomorphic
to some quotient ring of R .

$$\Rightarrow \frac{I+J}{J} \cong \frac{I}{\text{Ker } \phi}$$

$$\text{Ker } \phi = \{ i_k + J \in I+J \mid \phi(i_k + J) = I \}$$

$$\Rightarrow \text{Ker } \phi = \{ i_k + J \in I+J \mid i_k = I \}$$

$$\Rightarrow \text{Ker } \phi = \{ i_k + J \in J \mid i_k \in I \}$$

$$\Rightarrow \text{Ker } \phi = \{ i_k + J \in J \mid i_k + J \in I \}$$

$$\Rightarrow \text{Ker } \phi = \boxed{I \cap J}$$

$$\therefore \frac{I+J}{J} \cong \frac{I}{\underline{I \cap J}}$$

- Q.2 (d) Let $f(z) = \ln(1+z)$. Expand in a Taylor Series about $z = 0$. Determine the region of convergence of the series. (10)

The Taylor series expansion of a function about $z = z_0$ is given by:

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \dots + \frac{(z-z_0)^n}{n!} f^n(z_0)$$

Here $f(z) = \ln(1+z)$

$$f(z_0) = f(0) = \ln 1 = 0.$$

$$f'(z) = \frac{d}{dz} \ln(1+z) = \frac{1}{1+z} \Rightarrow f'(z_0) = 1$$

$$f''(z_0) = \left. \frac{-1}{(1+z)^2} \right|_{z=0} = -1$$

$$\therefore f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots + \frac{z^n (-1)^{n+1}}{n} + \dots$$

Any n^{th} term of $f(z) = \frac{z^n}{n} (-1)^{n+1}$.

Let $U_n = \frac{z^n}{n}$. Then $U_{n+1} = \frac{z^{n+1}}{n+1}$

$$\lim_{n \rightarrow \infty} \left| \frac{U_n}{U_{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{z^n/n}{z^{n+1}/(n+1)} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)}{n} \right| \left| \frac{1}{z} \right| = \frac{1}{|z|}$$

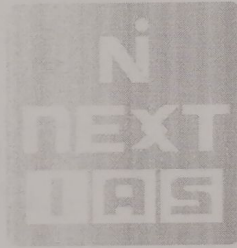
The series is convergent if $\lim_{n \rightarrow \infty} \frac{U_n}{U_{n+1}} > 1$

(By ratio test)

$\therefore$ The series is convg if $\left|\frac{1}{z}\right| > 1$

$$\text{or } \underline{\underline{|z| < 1}}$$

$\therefore$ The region of convergence is $|z| < 1$



Q.3 (a)

Suppose that f and g are two analytic functions on the set $\mathbb{C}$ of all complex numbers with

$$f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right) \text{ for } n = 1, 2, 3, \dots. \text{ Then show that } f(z) = g(z) \text{ for each } z \text{ in } \mathbb{C}. \quad (15)$$

Candidates
must not write
on this margin



Candidates
must not write
on this margin



Candidates
must not write
on this margin



Q.3 (b)

Let $f(x) \geq g(x)$ for every x in $[a, b]$ and f and g are both bounded and Riemann integrable on $[a, b]$. At a point $c \in [a, b]$, let f and g be continuous and $f(c) > g(c)$, then prove that

$$\int_a^b f(x) dx > \int_a^b g(x) dx \text{ and hence show that } -\frac{1}{2} < \int_{-\pi}^{\pi} \frac{x^3 \cos 5x}{2+x^2} dx < \frac{1}{2}. \quad (15)$$



Candidates
must not write
on this margin





Q.3 (c)

Five wagons are available at stations 1, 2, 3, 4 and 5. These are required at five stations I, II, III, IV and V. The mileages between various stations are given by the table below, How should the wagons be transported so as to minimize the total mileage covered?

(20)

	I	II	III	IV	V
1	10	5	9	18	11
2	13	9	6	12	14
3	3	2	4	4	5
4	18	9	12	1	15
5	11	6	14	19	10

Candidates
must not write
on this margin



Candidates
must not write
on this margin



Candidates
must not write
on this margin



Candidates
must not write
on this margin



Q.4 (a) Let $R = \mathbb{Z}[i] = \{a+ib \mid a, b \in \mathbb{Z}\}$. Let $M = \langle 2+i \rangle$ then show that M is maximal ideal of R .

$M = \langle 2+i \rangle$. Let $U = \langle c+id \rangle$ be another ideal such that $M \subseteq U \subseteq R$ (15)

$$\text{Since } M \subseteq U \Rightarrow \langle 2+i \rangle \subseteq \langle c+id \rangle$$

$$\Rightarrow (2+i) = (c+id)(e+if) \quad \text{--- (1)}$$

$$\Rightarrow c+id = \frac{2+i}{e+if} \Rightarrow c+id = \frac{2+i}{e+if} \times \frac{e-if}{e-if}$$

$$\Rightarrow c+id = \frac{(2e+f)}{(e^2+f^2)} + i \frac{(e-2f)}{(e^2+f^2)}$$

$\downarrow$
p (say)

$\downarrow$
q (say)

$\Rightarrow c+id = p+iq$ where p, q are rational numbers

Let $m = [p]$ if $\{p\} < \frac{1}{2}$

$m = [p] + 1$ if $\{p\} \geq \frac{1}{2}$

Now $p = (m + \delta)$ where $-\frac{1}{2} \leq \delta < \frac{1}{2}$

Similarly $q = n + \delta$ where $-\frac{1}{2} \leq \delta < \frac{1}{2}$

$$\begin{aligned} \text{now } (p+iq) &= (m+\delta) + i(n+\delta) \\ &= (m+in)(\delta+i\delta) \end{aligned}$$

$$\Rightarrow c+id = (m+in)(\delta+i\delta)$$

Taking conjugate on both sides of (1),

$$(2-i) = (c-id)(e-if) \quad \text{--- (2)}$$

Multiplying (1) & (2), $(2+i)(2-i) = (c^2+d^2)(e^2+f^2)$

$$\Rightarrow 5 = (c^2+d^2)(e^2+f^2)$$

Since 5 is prime, $c^2+d^2 = 1$ & $e^2+f^2 = 5$
(or) $c^2+d^2 = 5$ & $e^2+f^2 = 1$.

Case (I) $(c^2+d^2) = 1$ and $(e^2+f^2) = 5$.

$$c^2+d^2 = 1 \Rightarrow c=0, d=\pm 1 \text{ (or) } c=\pm 1, d=0$$

$$\Rightarrow c+id = \pm i \text{ or } \pm 1$$

$$\text{Then } \langle c+id \rangle = \langle \pm i \rangle = \langle \pm 1 \rangle = \mathbb{Z}[i]$$

$$\therefore \langle c+id \rangle \Rightarrow \mathbb{Z}[i] \subseteq U$$

$$\therefore U = \mathbb{Z}[i] \quad \text{--- (1)} \quad (\because U \subseteq \mathbb{Z}[i])$$

Case (II) $c^2+d^2 = 5$ & $e^2+f^2 = 1$

$$c^2+d^2 = 5 \text{ and } e+if = \pm i \text{ or } \pm 1.$$

$$\text{Then } (2+i) = (c+id)(e+if)$$

$$\Rightarrow (2+i) = (c+id)(\pm i) \text{ (or) } (2+i) = (c+id)(\pm 1)$$

$$\Rightarrow (c+id) = (2+i)(\mp i) \text{ (or)}$$

$$(c+id) = (2+i)(\pm 1)$$

$$\Rightarrow \langle c+id \rangle = \langle (2+i)(\pm i) \rangle \quad (\text{or})$$

$$\langle c+id \rangle = \langle (2+i)(\pm 1) \rangle$$

$$\Rightarrow \langle c+id \rangle \subseteq \langle 2+i \rangle \Rightarrow U \subseteq M$$

But as per our assumption $M \subseteq U$

$$\therefore M = U \quad \text{--- (2)}$$

$\therefore$ From ① & ②, if $M \subseteq U \subseteq Z$,

Then $U = Z$ or $M = U$. Hence,

M is a Maximal ideal.

Q.4 (b)

Prove by contour integration $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \ln 2$.

(15)

Consider the term $\int_{-\infty}^{\infty} \frac{\log(1+x^2)}{1+x^2} dx$:

$$\int_{-\infty}^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = 2 \int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx \quad (\because \text{The function is even})$$

Consider the function $\frac{\log(1+x)}{1+x^2}$:

$$\frac{\log(1+x)}{1+x^2} = \frac{\log(re^{i\theta})}{1+x^2} \quad \text{where } r = \sqrt{x^2+1} \text{ and } \theta = \tan^{-1}(1/x)$$

$$= \frac{\log r}{1+x^2} + i \frac{\log \theta}{1+x^2} = \frac{\log(\sqrt{1+x^2})}{1+x^2} + i \frac{\log \theta}{1+x^2}$$

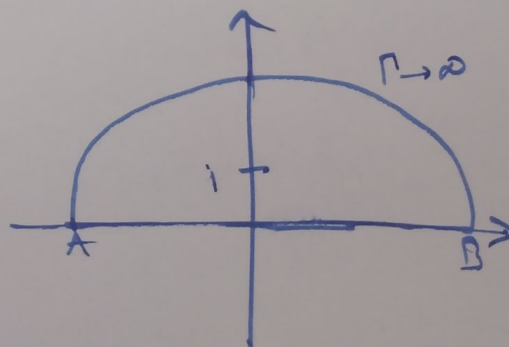
$$= \frac{1}{2} \frac{\log(1+x^2)}{(1+x^2)} + i \frac{\log \theta}{1+x^2}$$

$$\therefore \operatorname{Re} \left(\frac{\log(1+x)}{1+x^2} \right) = \frac{1}{2} \frac{\log(1+x^2)}{(1+x^2)}$$

Calculating $\int_{-\infty}^{\infty} \frac{\log(1+x)}{1+x^2} dx$: Taking the curve

as semicircle above real axis, with infinite radius:

$$\int_C \frac{\log(1+x)}{1+x^2} = \int_{AB} + \int_{BA}$$



$$\int_{BA} \frac{\log(i+x)}{1+x^2} = 0 \quad \text{as} \quad \lim_{x \rightarrow \infty} \frac{\log(i+x)}{1+x^2} = 0$$

$$\Rightarrow \int_C \frac{\log(i+x)}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{\log(i+x)}{1+x^2} dx$$

Poles for the function $\frac{\log(i+x)}{1+x^2} dx$ in C :

$$x = +i$$

$$\text{Residue at } x=i \Rightarrow \lim_{x \rightarrow i} (x-i) \frac{\log(i+x)}{1+x^2}$$

($\therefore$ Residue of pole at $z=a$ of order ' n ' is $\frac{1}{(n-1)!} \lim_{z \rightarrow a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$)

$$\therefore R_1 = \lim_{x \rightarrow i} \frac{\log(i+x)}{(x+i)} = \frac{1}{2i} \log(2i)$$

$$\Rightarrow R_1 = \frac{1}{2i} [\log(2) + \log(i)] = \frac{1}{2i} (\log 2 + \log e^{i\pi/2})$$

$$\Rightarrow R_1 = \frac{1}{2i} (\log 2 + i\pi/2)$$

$$\therefore \int_C \frac{\log(i+x)}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{\log(i+x)}{1+x^2} dx = 2\pi i \times R_1$$

(By Cauchy's residue theorem)

$$\therefore \int_{-\infty}^{\infty} \frac{\log(i+x)}{1+x^2} dx = \pi (\log 2 + i\pi/2)$$

Candidates
must not write
on this margin

$$\Rightarrow \operatorname{Re} \int_{-\infty}^{\infty} \left(\frac{\log(i+x)}{1+x^2} \right) dx = \pi \ln 2 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\log(1+x^2)}{1+x^2} dx$$

$$\Rightarrow \pi \ln 2 = \int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx$$

$$\therefore \int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \ln 2$$

Hence
proved.



Q.4 (c)

Solve the following linear Programming problem using Big M method:

(20)

Candidates must not write on this margin

$$\text{Maximise } Z = 4x_1 + 5x_2 + 2x_3$$

$$2x_1 + x_2 + x_3 \geq 10,$$

$$x_1 + 3x_2 + x_3 \leq 12$$

Subject to :

$$x_1 + x_2 + x_3 = 6$$

$$x_1, x_2, x_3 \geq 0$$

Modifying the constraints;

$$2x_1 + x_2 + x_3 - S_1 + R_1 = 10$$

$$x_1 + 3x_2 + x_3 + S_2 = 12$$

$$x_1 + x_2 + x_3 + R_2 = 6$$

$$\text{and Max } Z = 4x_1 + 5x_2 + 2x_3 + 0S_1 + 0S_2 - MR_1 - MR_2$$

Writing the Big M Table :

	x_1	x_2	x_3	S_1	S_2	R_1	R_2	soln	Ratio
C_j	4	5	2	0	0	-M	-M		
-M R_1	2	1	1	-1	0	1	0	10	5 ←
0 S_2	1	3	1	0	1	0	0	12	12
-M R_2	1	1	1	0	0	0	1	6	6
Z_j	-3M	-2M	-2M	M	0	-M	-M	-6M	
$C_j - Z_j$	4+3M ↑	5+2M	2+2M	-M	0	0	0		
4 x_1	1	1/2	1/2	-1/2	0	1/2	0	5	10
0 S_2	0	5/2	1/2	1/2	1	-1/2	0	7	14/5
-M R_2	0	1/2	1/2	1/2	0	-1/2	0	1	2 ←
Z_j	4	2-M/2	2-M/2	-2-M/2	0	M+M/2	0	20-M	
$C_j - Z_j$	0	5+M/2 ↑	M/2	2+M/2	0	-2M-M/2	-M		

|| P.T.O

	x_1	x_2	x_3	s_1	s_2	R_1	R_2	Soln Ratio
$4x_1$	1	0	0	-1	0	0	0	4
$0s_2$	0	0	-2	2	1	2	0	2
$5x_2$	0	1	1	1	0	-1	0	2
Z_j	4	5	5	1	0	-5	0	26 = 2
$C_j - Z_j$	0	0	-3	-1	0	-11+5	-11	

$\therefore$ All $C_j - Z_j \leq 0$. Hence,
optimality is reached.

Hence Max $Z = 26$ where,

$x_1 = 4, x_2 = 2, x_3 = 0$

Candidates
must not write
on this margin



Candidates
must not write
on this margin



SECTION-B

Q.5 (a)

Construct a partial differential equation of all surfaces of revolution having the z axis as the axis of rotation. (10)

The function is given

by $z = f(r)$ where $r = \sqrt{x^2 + y^2}$

$$r^2 = x^2 + y^2$$

$$\Rightarrow 2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = x/r$$

$$\text{Similarly, } \frac{\partial r}{\partial y} = y/r$$

Differentiating $z = f(r)$ w.r.t x ,

$$\frac{\partial z}{\partial x} = p = \frac{\partial}{\partial x} (f(r)) = f'(r) \times \frac{\partial r}{\partial x}$$

$$\Rightarrow p = f'(r) \times \frac{x}{r} \quad \text{--- (1)}$$

Similarly differentiating $z = f(r)$ w.r.t y ,

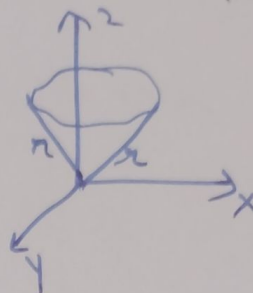
$$\frac{\partial z}{\partial y} = q = \frac{\partial}{\partial y} (f(r))$$

$$\Rightarrow q = f'(r) \times \frac{y}{r} \quad \text{--- (2)}$$

$$\text{Now (1)/(2)} \Rightarrow p/q = x/y$$

$$\Rightarrow \boxed{py - qx = 0} \quad \text{or} \quad \boxed{y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 0}$$

is the reqd PDE.



Candidates
must not write
on this margin



Q.5 (b)

Find the order of convergence of Regula-Falsi Method.

(10)

Candidates
must not write
on this marginOrder of convergence = p (say) $|\epsilon_{i+1}| = K |\epsilon_i|^p$ where ϵ_i is the error
in i^{th} iteration.

For Regula falsi Method:

$$x_i = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$\text{Order of convg} = \frac{\sqrt{5}+1}{2} \approx 1.618$$

Candidates
must not write
on this margin



Q.5 (c)

Find the values of the two valued variables A, B, C and D by solving the set of simultaneous equations $A' + AB = 0, AB = AC, AB + AC' + CD = C'D$. (10)

Candidates must not write on this margin

Given: $A' + AB = 0$ — (1)

From (1), it is clearly evident that

$$A' = 0 \text{ and } AB = 0 \quad (\because x + y = 0 \Rightarrow x = y = 0)$$

$$\Rightarrow A' = 0 \Rightarrow \underline{A = 1} \text{ — (2)}$$

$$A = 1 \text{ and } AB = 0 \Rightarrow \underline{B = 0} \text{ — (3)}$$

$$AB = AC :$$

$$AB = 0 = AC;$$

$$AC = 0 \Rightarrow (1) \cdot C = 0 \Rightarrow \underline{C = 0} \text{ — (4)}$$

Substituting these values in

$$AB + AC' + CD = C'D;$$

$$0 + 1 + 0 = C'D = (1)(D)$$

$$\Rightarrow \underline{1 = (1)(D)} \Rightarrow \underline{D = 1} \text{ — (5)}$$

$$\therefore \underline{A = 1, B = 0, C = 0, D = 1}$$

are the reqd values of the eqns.



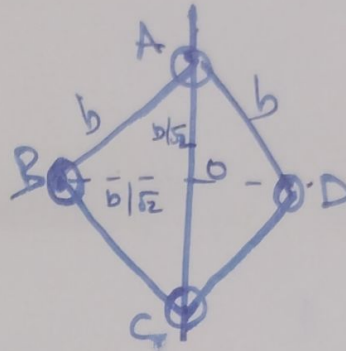
Q.5 (d)

Four solid sphere A, B, C and D, each of mass m and radius a , are placed with their centres on the four corners of a square of side b . Calculate the moment of inertia of the system about a diagonal of the square (10)

Candidates must not write on this margin

To find: $\underline{I_A}$

The MoI of spheres at A & C are



$$I_A = I_C = \frac{2}{5} ma^2.$$

By parallel axis theorem, I_B and I_D are found out by:

$$I_B = \frac{2}{5} ma^2 + m \times OB^2 = \frac{2}{5} ma^2 + m \left(\frac{b}{\sqrt{2}} \right)^2$$

$$\Rightarrow I_B = \frac{2}{5} ma^2 + \frac{mb^2}{2}.$$

$$\text{Similarly } I_D = \frac{2}{5} ma^2 + \frac{mb^2}{2}.$$

$$\text{Total MoI} = I_A + I_B + I_C + I_D$$

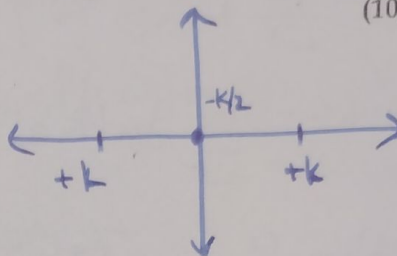
$$= \frac{2}{5} ma^2 + \frac{2}{5} ma^2 + \left(\frac{2}{5} ma^2 + \frac{mb^2}{2} \right) + \left(\frac{2}{5} ma^2 + \frac{mb^2}{2} \right)$$

$$\therefore I = \frac{8}{5} ma^2 + mb^2$$

Q.5 (e)

Two point vortices each of strength k are situated at $(\pm a, 0)$ and a point vortex of strength $-k/2$ is situated at the origin. Show that the fluid motion is stationary and find the equations of streamlines. (10)

For a vortex of strength k ,
the complex potential
is given by $w = \frac{ik}{2\pi} \ln(z - z_1)$



For the vortex at $z = a$, the complex potential
on it is $w_a = \frac{ik}{2\pi} \ln(z + a) - \frac{ik}{4\pi} \ln(z)$

$$\left. \frac{dw}{dz} \right|_{z=a} = \frac{ik}{2\pi} \times \frac{1}{(z+a)} - \frac{ik}{4\pi} \times \frac{1}{z} \Big|_{z=a}$$

$$\frac{dw}{dz} = \frac{ik}{4\pi a} - \frac{ik}{4\pi a} = \underline{\underline{0}}$$

$\therefore$ The vortex is stationary.

By symmetry; $\left. \frac{dw}{dz} \right|_{z=-a} = 0$

Also, at origin, $w_0 = \frac{ik}{2\pi} \ln(z-a) + \frac{ik}{2\pi} \ln(z+a)$

$$\left. \frac{dw_0}{dz} \right|_{z=0} = \frac{ik}{2\pi} \frac{1}{z-a} + \frac{ik}{2\pi} \frac{1}{z+a} \Big|_{z=0} = 0$$

$$\therefore \left. \frac{dw_0}{dz} \right|_{z=0} = 0$$

Hence the 3 vortices are stable.

$\therefore$ Fluid motion is stationary.

Finding eqn of streamlines.

$$w = \frac{ik}{2\pi} \ln(z-a) + \frac{ik}{2\pi} \ln(z+a) - \frac{ik}{4\pi} \ln(z)$$

$$\Rightarrow w = \frac{ik}{2\pi} \ln(R_1 e^{i\theta_1}) + \frac{ik}{2\pi} \ln(R_2 e^{i\theta_2}) - \frac{ik}{4\pi} \ln(R_3 e^{i\theta_3})$$

$$\text{Here } R_1 = \sqrt{(x-a)^2 + y^2}; R_2 = \sqrt{(x+a)^2 + y^2}$$

$$R_3 = \sqrt{x^2 + y^2}$$

$$w = \phi + i\psi \Rightarrow \underline{\psi = \text{Im}(w)}$$

$$\therefore \psi = \frac{K}{2\pi} \ln(R_1) + \frac{K}{2\pi} \ln(R_2) - \frac{K}{4\pi} \ln(R_3)$$

$$\Rightarrow \psi = \frac{K}{2\pi} \ln(\sqrt{(x-a)^2 + y^2}) + \frac{K}{2\pi} \ln(\sqrt{(x+a)^2 + y^2}) - \frac{K}{4\pi} \ln(\sqrt{x^2 + y^2})$$

$$\Rightarrow \psi = \frac{K}{4\pi} \ln \left(\frac{[(x-a)^2 + y^2][(x+a)^2 + y^2]}{(x^2 + y^2)^{1/2}} \right)$$

Streamlines given by $\psi = C$

$$\Rightarrow \frac{K}{4\pi} \ln \left(\frac{[(x-a)^2 + y^2][(x+a)^2 + y^2]}{(x^2 + y^2)^{1/2}} \right) = C$$

$$\Rightarrow [(x-a)^2 + y^2][(x+a)^2 + y^2] = P (x^2 + y^2)^{1/2}$$

$$\Rightarrow \boxed{\frac{[(x-a)^2 + y^2]^2 [(x+a)^2 + y^2]^2}{(x^2 + y^2)} = P}$$

is the reqd eqn of streamlines.

Q.6 (a) Find the integral surface of the linear partial differential equation

$$x(y^2+z)\frac{\partial z}{\partial x} - y(x^2+z)\frac{\partial z}{\partial y} = (x^2-y^2)z \text{ through the straight line } x+y=0, z=1. \quad (18)$$

Given eqn is of the form $pP(x,y,z) + qQ(x,y,z) = R(x,y,z)$

Writing the Lagrange eqns:

$$P = x(y^2+z), \quad Q = -y(x^2+z);$$

$$R = (x^2-y^2)z$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

$$= \frac{x dx + y dy - dz}{0}$$

$\Rightarrow x dx + y dy - dz = 0$. Integrating,

$$\boxed{x^2 + y^2 - z = C_1} \quad \text{--- (i)}$$

$$\text{Also, } \frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

$$= \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \text{ Integrating,}$$

$$\ln x + \ln y + \ln z = C' \Rightarrow \boxed{xyz = C_2} \quad (2)$$

Now, the surface passes through $x+y=0, z=1$

$$\text{When } z=1, \quad xy = C_2, \quad x^2 + y^2 = 1 + C_1$$

$$(x+y)=0 \Rightarrow (x+y)^2 = 0 \Rightarrow x^2 + y^2 + 2xy = 0$$

$$\Rightarrow (1 + C_1) + 2(C_2) = 0$$

$$\Rightarrow C_1 + 2C_2 = -1 \quad (\text{or}) \quad C_1 + 2C_2 + 1 = 0.$$

$$\Rightarrow (x^2 + y^2 - z) + 2(xyz) + 1 = 0$$

$\Rightarrow$ The required surface is

$$\boxed{x^2 + y^2 - z + 2xyz + 1 = 0}$$



Candidates
must not write
on this margin



Q.6 (b)

Given $f(1) = 4, f(2) = 5, f(7) = 5$ and $f(8) = 4$. Find the value of $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum. (15)

Calculating the polynomial by Lagrange's interpolation:

	x_0	x_1	x_2	x_3
x	1	2	7	8
$f(x)$	4	5	5	4
	y_0	y_1	y_2	y_3

The formula for the interpolation is:

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$\Rightarrow f(x) = \frac{(x-2)(x-7)(x-8)}{(1-2)(1-7)(1-8)} (4) + \frac{(x-1)(x-7)(x-8)}{(2-1)(2-7)(2-8)} (5)$$

$$+ \frac{(x-1)(x-2)(x-8)}{(7-1)(7-2)(7-8)} (5) + \frac{(x-1)(x-2)(x-7)}{(8-1)(8-2)(8-7)} (4)$$

$$\Rightarrow f(x) = \frac{(x-2)(x-7)(x-8)}{(-1)(-6)(-7)} \left(-\frac{4}{42}\right) + \frac{(x-1)(x-7)(x-8)}{(-1)(-5)(-6)} \left(\frac{5}{30}\right)$$

$$+ \frac{(x-1)(x-2)(x-8)}{(-6)(-5)(-1)} \left(-\frac{5}{30}\right) + \frac{(x-1)(x-2)(x-7)}{(-1)(-6)(-1)} \left(\frac{4}{42}\right)$$

$$\Rightarrow f(x) = \left(\frac{4}{42}\right) (x-7)(x-8) \left[(8-x) + (x-1) \right]$$

$$+ \left(\frac{5}{30}\right) (x-1)(x-8) \left[x-7 + 2-x \right]$$

$$\Rightarrow f(x) = \frac{4}{12} (x-1)(x-2) (7) + \frac{5}{30} (x-1)(x-2) (-5)$$

$$\Rightarrow f(x) = \frac{2}{3} (x^2 - 9x + 14) - \frac{5}{6} (x^2 - 9x + 8)$$

$$\Rightarrow \boxed{f(x) = -\frac{1}{6}x^2 + \frac{3}{2}x + \frac{8}{3}}$$
 is the required polynomial

$$f(b) = -\frac{1}{6}(b)^2 + \frac{3}{2}(b) + \frac{8}{3} = 17/3$$

$$\Rightarrow \boxed{f(b) = 17/3}$$

For max/min value, $f'(x) = 0$.

$$f'(x) = 0 \Rightarrow -\frac{2x}{6} + \frac{3}{2} = 0$$

$$\Rightarrow -\frac{x}{3} = -3/2 \Rightarrow \boxed{x = 9/2}$$

$$f''(x) \Big|_{x=9/2} = -1/3 < 0$$

$\Rightarrow f(x)$ has a maximum at $x = 9/2$

$$\text{Max value of } f(x) = 145/24.$$

- Q.6 (c) For a dynamical system $T = \frac{1}{2}\{(1+2k)\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2\}$, $V = \frac{n^2}{2}\{(1+k)\theta^2 + \phi^2\}$, where θ, ϕ are coordinates and n, k are positive constants, write down the Lagrange's equations of motion and deduce that $(\theta - \phi) + n^2 \frac{1+k}{k}(\theta - \phi) = 0$. Further show that if $\theta = \phi, \dot{\theta} = \dot{\phi}$ at $t = 0$ then $\theta = \phi$ for all t . (17)

We know that $L = T - V$

$$\Rightarrow L = \frac{1}{2}((1+2k)\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2) - \frac{1}{2}((1+k)n^2\theta^2 + n^2\phi^2)$$

$$\Rightarrow L = \frac{1}{2}[(1+2k)\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2 - (1+k)n^2\theta^2 - n^2\phi^2]$$

The Lagrange's eqns of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}[2(1+2k)\dot{\theta} + 2\dot{\phi}]\right) - \left[\frac{1}{2}(-2(1+k)n^2\theta)\right] = 0$$

$$\Rightarrow \frac{d}{dt}((1+2k)\dot{\theta} + \dot{\phi}) + n^2(1+k)\theta = 0$$

$$\Rightarrow (1+2k)\ddot{\theta} + \ddot{\phi} + n^2(1+k)\theta = 0 \quad \text{--- (1)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}(2\dot{\theta} + 2\dot{\phi})\right) - \left(\frac{1}{2}(-2n^2\phi)\right) = 0$$

$$\Rightarrow \ddot{\theta} + \ddot{\phi} + n^2\phi = 0 \quad \text{--- (2)}$$

~~$$(1) - (2) \Rightarrow 2k \ddot{\theta} + n^2(1+k)\theta - n^2\phi = 0$$~~

~~$$\Rightarrow 2k \ddot{\theta} + n^2(1+k)\theta = n^2\phi$$~~

~~Integrating by multiplying θ on both sides~~

~~$$\Rightarrow \int 2k \ddot{\theta} \theta + n^2(1+k) \int \dot{\theta} \theta = \int n^2 \phi \theta$$~~

~~$$\Rightarrow 2k \dot{\theta}^2 + n^2(1+k) \frac{\theta^2}{2} = n^2 \phi \theta$$~~

Multiplying (2) with $(1+k)$ on both sides :

$$\ddot{\theta}(1+k) + \ddot{\phi}(1+k) + n^2(1+k)\phi = 0 \quad (3)$$

$$(1) - (3)$$

$$\Rightarrow \ddot{\theta}(1+2k-1-k) + \ddot{\phi}(1-1-k) + n^2(1+k)(\theta-\phi) = 0$$

$$\Rightarrow \ddot{\theta}(k) + \ddot{\phi}(-k) + n^2(1+k)(\theta-\phi) = 0$$

$$\Rightarrow \boxed{(\ddot{\theta} - \ddot{\phi}) + n^2 \frac{(1+k)}{k} (\theta - \phi) = 0} \text{ is the required eqn.}$$

$$\hookrightarrow (4)$$

Let $\theta - \phi = p$, then $\ddot{\theta} - \ddot{\phi} = \ddot{p}$, $\dot{\theta} - \dot{\phi} = \dot{p}$

$$\text{From (4), } (\ddot{\theta} - \ddot{\phi}) = -\frac{n^2(1+k)}{k} (\theta - \phi)$$

$$\Rightarrow \ddot{p} = -\frac{n^2(1+k)}{k} p \cdot \text{Multiplying}$$

Both sides with $2 \dot{p}$ and integrating

$$\Rightarrow \int 2 \dot{p} \ddot{p} = -\frac{2n^2(1+k)}{k} \int \dot{p} p$$

$$\Rightarrow \int d\left(\frac{dp}{dt}\right)^2 = \int \frac{-2n^2}{k} (1+k) p dp$$

$$\Rightarrow \left(\frac{dp}{dt}\right)^2 + C = -\frac{2n^2}{k} (1+k) \left(\frac{p^2}{2}\right) = -\frac{n^2(1+k)}{k} p^2$$

$$\Rightarrow \left(\frac{dp}{dt}\right)^2 + C = -\frac{n^2(1+k)}{k} p^2. \text{ At } t=0,$$

$$\frac{dp}{dt} = 0 \text{ and } p=0 \quad (\because \theta-\phi=0=\dot{\theta}-\dot{\phi})$$

$$\Rightarrow C=0.$$

$$\therefore \left(\frac{dp}{dt}\right)^2 = -\frac{n^2(1+k)}{k} p^2$$

$$\Rightarrow \left(\frac{dp}{dt}\right) = p \sqrt{\frac{n^2(1+k)}{k}}$$

$$\Rightarrow \frac{dp}{p} = \sqrt{\frac{n^2(1+k)}{k}} dt: \text{ Integrating on}$$

$$\text{both sides, } \int \frac{dp}{p} = \int \sqrt{\frac{n^2(1+k)}{k}} dt.$$

$$\Rightarrow \ln p + C = \sqrt{\frac{n^2(1+k)}{k}} (t) \quad \text{---}$$

Candidates
must not write
on this margin



Candidates
must not write
on this margin

