

Online

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| Q. No.      | Page No. | Max. Marks | Marks | Total | Signature |
|-------------|----------|------------|-------|-------|-----------|
| 1. (a)      | 4        | 10         |       |       |           |
| 1. (b)      | 6        | 10         |       |       |           |
| 1. (c)      | 8        | 10         |       |       |           |
| 1. (d)      | 10       | 10         |       |       |           |
| 1. (e)      | 12       | 10         |       |       |           |
| 2. (a)      | 14       | 20         |       |       |           |
| 2. (b)      | 18       | 15         |       |       |           |
| 2. (c)      | 21       | 15         |       |       |           |
| 3. (a)      | 24       | 15         |       |       |           |
| 3. (b)      | 27       | 15         |       |       |           |
| 3. (c)      | 30       | 20         |       |       |           |
| 4. (a)      | 34       | 20         |       |       |           |
| 4. (b)      | 38       | 15         |       |       |           |
| 4. (c)      | 41       | 15         |       |       |           |
| 5. (a)      | 44       | 10         |       |       |           |
| 5. (b)      | 46       | 10         |       |       |           |
| 5. (c)      | 48       | 10         |       |       |           |
| 5. (d)      | 50       | 10         |       |       |           |
| 5. (e)      | 52       | 10         |       |       |           |
| 6. (a)      | 54       | 15         |       |       |           |
| 6. (b)      | 57       | 20         |       |       |           |
| 6. (c)      | 61       | 15         |       |       |           |
| 7. (a)      | 64       | 20         |       |       |           |
| 7. (b)      | 68       | 15         |       |       |           |
| 7. (c)      | 71       | 15         |       |       |           |
| 8. (a)      | 74       | 20         |       |       |           |
| 8. (b)      | 78       | 15         |       |       |           |
| 8. (c)      | 81       | 15         |       |       |           |
| Grand Total |          |            |       |       |           |

Remarks

Observations:

The following observations were made during the experiment:  
 1. The reaction was very rapid.  
 2. The color of the solution changed from colorless to yellow.  
 3. The temperature of the solution increased.  
 4. The reaction was exothermic.  
 5. The reaction was reversible.  
 6. The reaction was catalyzed by the presence of the substance.  
 7. The reaction was inhibited by the presence of the substance.  
 8. The reaction was accelerated by the presence of the substance.  
 9. The reaction was retarded by the presence of the substance.  
 10. The reaction was completed within a short time.

## SECTION-A

- Q.1 (a) Let  $U$  and  $W$  be subspaces of vector space  $V$  and  $x, y \in V$ . Then prove that  
 $x+U \subseteq y+W$  iff  $U \subseteq W$  and  $x-y \in W$ . (10)

$$\text{Let } x+U \subseteq y+W ; x, y \in V$$

$$\text{To prove } U \subseteq W ; x-y \in W$$

$$\text{let } u_i \in U$$

$$x+u_i = y+w_j \quad \forall u_i \in U$$

$$u_i = (y-x) + w_j \quad \text{--- i}$$

$$0 \in U \quad (\because U \text{ is subspace i.e.})$$

$$x+0 = y+w_k$$

$$w_k = x-y$$

$$\therefore w_k \in W$$

$$\Rightarrow x-y \in W$$

$$\Rightarrow y-x \in W \quad \text{--- ii} \quad (W \text{ is subspace})$$

from i, ii

$$RHS \in W$$

$$\Rightarrow u_i \in W \quad \forall u_i \in U$$

$$\Rightarrow U \subseteq W$$

Let  $U \subseteq W$  and  $x - y \in W$

To prove  $x + U \subseteq y + W$ .

$$x - y \in W$$

$$\Rightarrow x - y + W = W$$

$$x + W = y + W.$$

$$U \subseteq W$$

$$\Rightarrow x + U \subseteq x + W$$

$$\Rightarrow x + U \subseteq y + W.$$

Show that 0 is characteristic root of matrix iff the matrix is singular.

(10)

Candidate must write on this

let 0 is characteristic root

$$\Rightarrow \lambda = 0 \quad |A - \lambda I| = 0$$

$$|A - 0I| = 0$$

$$|A| = 0$$

$\Rightarrow A$  is singular

let  $A$  is singular

$$\Rightarrow |A| = 0$$

$$\Rightarrow |A - 0I| = 0$$

$$|A - \lambda I| = 0$$

on comparing  $\lambda = 0$

$\Rightarrow 0$  is characteristic root

Q.1 (c)

Evaluate  $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}$  as  $x \rightarrow 0$ .

(10)

$$L = \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}$$

$$\log L = \lim_{x \rightarrow 0} \frac{1}{x^2} \log \cos x$$

$$\left( \frac{0}{0} \right)$$

$$= \frac{\frac{1}{\cos x} (-\sin x)}{2x}$$

$$\log L = \lim_{x \rightarrow 0} \frac{1}{2 \cos x} \left( \frac{\sin x}{x} \right)$$

$$\log L = \frac{-1}{2} (1)$$

$$L = e^{-1/2} = \frac{1}{\sqrt{e}}$$

Q.1 (d) Test the convergence of  $\int_0^1 \frac{\sin\left(\frac{1}{x}\right)}{\sqrt{x}} dx$

(10)

at 1 integrand is proper.

at 0  $\rightarrow$

$$f(x) = \frac{\sin\left(\frac{1}{x}\right)}{\sqrt{x}}$$

$$g(x) = \frac{1}{x^u}$$

$$\lim_{x \rightarrow 0} \frac{f}{g} = \lim_{x \rightarrow 0} x^{u-\frac{1}{2}} \sin\left(\frac{1}{x}\right)$$

$$= 0 \text{ (quantity between -1 and 1)}$$

$$\left( \text{if } u - \frac{1}{2} > 0 \right)$$

i.e.  $u > \frac{1}{2}$   
If  $u > \frac{1}{2}$ ,  $f, g$  converge or diverge together.

$$\text{let } u = \frac{3}{4}$$

$g(x)$  converges  $\Rightarrow$

$\Rightarrow f$  converges.

Q.1 (e)

Find the equation of the cylinder whose generators touch the sphere  $x^2 + y^2 + z^2 = 9$  and are perpendicular to the plane  $x - y - 3z = 5$ . (10)

$$\frac{x - x_1}{1} = \frac{y - y_1}{-1} = \frac{z - z_1}{-3} = r$$

$$(x, y, z) = (r + x_1, -r + y_1, -3r + z_1)$$

touch ~~the~~ on sphere.

$$(r + x_1)^2 + (-r + y_1)^2 + (-3r + z_1)^2 = 9$$

$$11r^2 + r(2x_1 - 2y_1 - 6z_1) + x_1^2 + y_1^2 + z_1^2 - 9 = 0$$

it has equal roots.

$$(x_1 - y_1 - 3z_1)^2 - 11(x_1^2 + y_1^2 + z_1^2 - 9) = 0$$

$$\cancel{10x_1^2 - 10y_1^2}$$

so equation of cylinder is

$$(x + y + 3z)^2 - 11(x^2 + y^2 + z^2 - 9) = 0.$$

- Q.4 (a) Show that the set  $\{(1, 2, 0, 3), (4, 0, 5, 8), (8, 1, 5, 6)\}$  is linearly independent in  $\mathbb{R}^4$ . The vectors form a basis for a three-dimensional subspace  $V$  of  $\mathbb{R}^4$ . Construct an orthonormal basis for  $V$ . (20)

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 4 & 0 & 5 & 8 \\ 8 & 1 & 5 & 6 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 4R_1 \\ R_3 \rightarrow R_3 - 8R_1 \end{array} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -8 & 5 & -4 \\ 0 & -15 & 5 & -18 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{15}{8}R_2 \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -8 & 5 & -4 \\ 0 & 0 & -\frac{35}{8} & -\frac{51}{2} \end{bmatrix}$$

$$\rho(A) = 3$$

$$\Rightarrow \dim(V) = 3.$$

(15)

Q.4 (b) Find the asymptotes of  $x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0$ .

No asymptote parallel to axes.

$$x = 1; y = m$$

$$1 + 2m - m^2 - 2m^3 + m - m^2 - 1 = 0$$

$$2m^3 + 2m^2 - 3m = 0$$

$$m(2m^2 + 2m - 3) = 0$$

$$2m(m^2 + m - \frac{3}{2}) = 0$$

$$\Rightarrow m = 0.$$

$$c = - \frac{\phi_3(0)}{\phi_3'(0)} = \frac{0}{0}$$

$$\frac{c^2}{2} \phi_3''(0) + c \phi_2'(0) + \phi_1(0) = 0$$

$$\frac{c^2}{2} (0) + c(0) +$$

$$\text{put } y = mx + c.$$

$$x^3 + 2x^2(mx + c) - x(mx + c)^2 - 2(mx + c)^3 + x(mx + c) - (mx + c)^2 - 1 = 0$$

$$x^3 [1 + 2m - m^2 - 2m^3] + x^2 [2c - 2mc - 6m^2c + m^3]$$

$$+x[-c^2 - 6c^2m + c - 2mc] - 2c^3 - c^2 - 1 = 0$$

Dividing by  $x^3$  &  $x \rightarrow \infty$

$$\Rightarrow \begin{aligned} 1 + 2m - m^2 - 2m^3 &= 0 \\ 2m^3 + m^2 - 2m - 1 &= 0 \\ (m-1)(2m^2 + 3m + 1) &= 0 \end{aligned}$$

$$(m-1)(m+1)(2m+1) = 0$$

$$m = 1, -1, -\frac{1}{2}$$

$$\text{Now } 2c - 2mc - 6m^2c + m - m^2 = 0$$

$$m=1 \Rightarrow c=0$$

$$m=-1 \Rightarrow c=-1$$

$$m=-\frac{1}{2} \Rightarrow c=\frac{1}{2}$$

$$\Rightarrow y=x; y=-x-1; y=\frac{x}{2}+\frac{1}{2}$$

Doubt  $\rightarrow$  Why the approach which is ~~crossed~~ in box not works?

- c) Prove that the centres of the spheres which touch the lines  $y = mx, z = c$  and  $y = -mx, z = -c$  must lie on the conicoid  $mxy + c(1 + m^2)z = 0$ . (15)

Let  $\alpha, \beta, \gamma$  be centre of sphere,  $r$  is radius  
 $(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2$

$$L_1: \frac{x}{1} = \frac{y}{m} = \frac{z - c}{0}$$

$$L_2: \frac{x}{1} = \frac{y}{-m} = \frac{z + c}{0}$$

Distance of  $(\alpha, \beta, \gamma)$  from  $L_1, L_2$  is  $r$  (radius).

$$r^2(1^2 + m^2 + 0^2) = \left\{ [m(\alpha) - 1(\beta)]^2 + [0(\beta) - m(\gamma - c)]^2 + [0(\alpha) - (\gamma - c)]^2 \right\}$$

$$r^2(m^2 + 1) = \left\{ (m\alpha - \beta)^2 + m^2(\gamma - c)^2 + (\gamma - c)^2 \right\}$$

$$r^2(m^2 + 1) = \left\{ (m\alpha + \beta)^2 + m^2(\gamma + c)^2 + (\gamma + c)^2 \right\}$$

Subtracting

$$(m\alpha + \beta)^2 + m^2(\gamma + c)^2 + (\gamma + c)^2 = (m\alpha - \beta)^2 + m^2(\gamma - c)^2 + (\gamma - c)^2$$

$$\Rightarrow 2\beta(2m\alpha) + 4m^2\gamma c + 4\gamma c = 0$$

$$m\alpha\beta + cy(1+m^2) = 0$$

$$m\alpha\gamma + cz(1+m^2) = 0$$

## SECTION-B

- Q.5 (a) Find the Wronskian of the set of functions  $\{3x^3, |x^3|\}$  on the interval  $[-1, 1]$  and determine whether the set is linearly dependent on  $[-1, 1]$ . (10)

$$f_1(x) = 3x^3; \quad f_1'(x) = 9x^2$$

$$f_2(x) = |x^3| = \begin{cases} -x^3 & [-1, 0) \\ +x^3 & [0, 1] \end{cases}$$

$$\text{LHD } \lim_{h \rightarrow 0^+} \frac{f_2(0-h) - f_2(0)}{-h} = \lim_{h \rightarrow 0^+} \frac{-h^3}{-h} = 0$$

$$\text{RHD } \lim_{h \rightarrow 0^+} \frac{f_2(0+h) - f_2(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h^3}{h} = 0$$

$$W(f_1, f_2) = \begin{cases} \begin{vmatrix} 3x^3 & -x^3 \\ 9x^2 & -3x^2 \end{vmatrix} & x \in [-1, 0) \\ \begin{vmatrix} 3x^3 & x^3 \\ 9x^2 & 3x^2 \end{vmatrix} & x \in [0, 1] \end{cases}$$

$$= \begin{cases} 0 & x \in [-1, 0) \\ 0 & x \in [0, 1] \end{cases}$$

$$\Rightarrow W(f_1, f_2) = 0 \quad \forall x \in [-1, 1]$$

$\Rightarrow$  set is linearly dependent  
on  $[-1, 1]$

Q.5 (b) The necessary and sufficient condition for  $f(t)$  to be constant is  $\frac{df}{dt} = 0$ . (10)

Necessary.

let  $f(t)$  is constant.

$$\text{i.e. } f(t) = C$$

$$\Rightarrow \frac{df(t)}{dt} = \frac{dC}{dt} = 0$$

$$\Rightarrow \frac{df(t)}{dt} = 0$$

Sufficient.

$$\text{let } \frac{df}{dt} = 0$$

$\Rightarrow$  ~~if~~ Integrating both sides w.r.t  $t$  we get

$$f(t) = C.$$

i.e.  $f$  is constant.

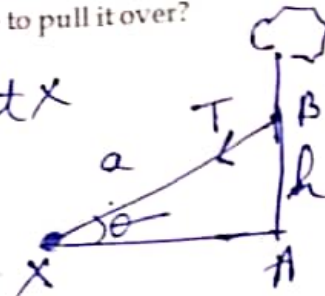
Q.5 (c)

At what point of a tree must one of a rope of given length 'a' be attached so that a man pulling at the other end may exert greatest force to pull it over? (10)

Let man be point sized at X

AB is tree

B is varying tied point.



$$T_x = T \cos \theta$$

$$T_y = T \sin \theta$$

~~for~~  $T_y$  will be balanced  
by reaction from ground  
to tree

So pulling force is  $T_x$

max  $T_x$  when  $\cos \theta$  is max

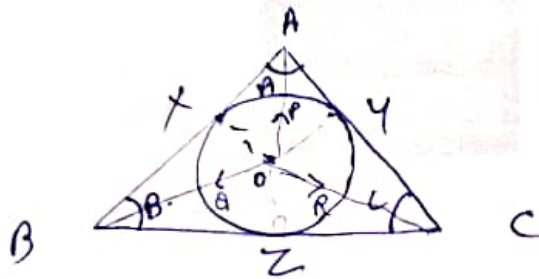
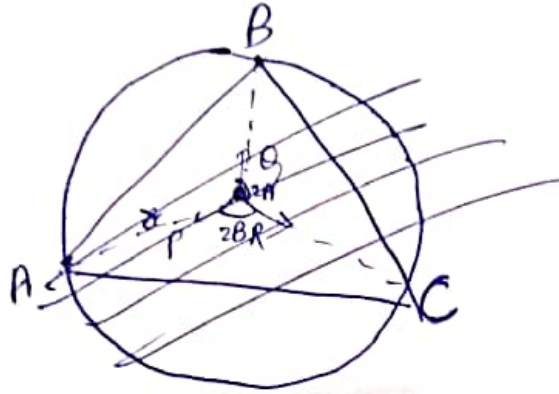
$$\text{i.e. } \theta = 0.$$

i.e. to be attached at bottom

2.5 (d)

A particle is placed at the centre of a circle and is acted on by three forces  $P, Q$  and  $R$  tending towards the angular points  $A, B$  and  $C$  of a circumscribed triangle. Prove that when

the forces are in equilibrium then  $\frac{P}{\cos \frac{A}{2}} = \frac{Q}{\cos \frac{B}{2}} = \frac{R}{\cos \frac{C}{2}}$ . (10)



$$\frac{P}{\sin 2A} = \frac{Q}{\sin 2B}$$

$$\frac{P}{\sin \left( \frac{B+C}{2} \right)} = \frac{Q}{\sin \left( \frac{A+C}{2} \right)}$$

Consider  $\triangle OXA$  and  $\triangle OYA$

$$\angle OXA = \angle OYA = 90^\circ$$

$$AO = AO = \text{hypotenuse}$$

$$OX = OY = \text{radius}$$

$$\Rightarrow \triangle OXA \sim \triangle OYA$$

$$\Rightarrow \angle XAO = \angle YAO = A/2$$

Candidates  
must not write  
on this margin

Summary  $\angle OBZ = B/2$ ;  $\angle OZC = C/2$

$$\angle ZO B = 90 - B/2 \quad ; \quad \angle COZ = 90 - \frac{C}{2}$$

$$\Rightarrow \angle BOC = 180 - \frac{B+C}{2} - i$$

~~NO~~ Summary

$$\angle BOA = 180 - \frac{B+A}{2} \quad -ii$$

$$\angle AOC = 180 - \frac{A+C}{2} \quad -iii$$

$$i + ii + iii = 360 = 540 - (A+B+C)$$

$$\Rightarrow A+B+C = 180.$$

$$\Rightarrow \angle BOC = 180 - \frac{180 - A}{2}$$

$$\Rightarrow = \frac{180 + A}{2} = 90 + \frac{A}{2}$$

$$\Rightarrow \angle BOA = 90 + \frac{C}{2} \quad ; \quad \angle COA = 90 + \frac{B}{2}$$

Lami formula

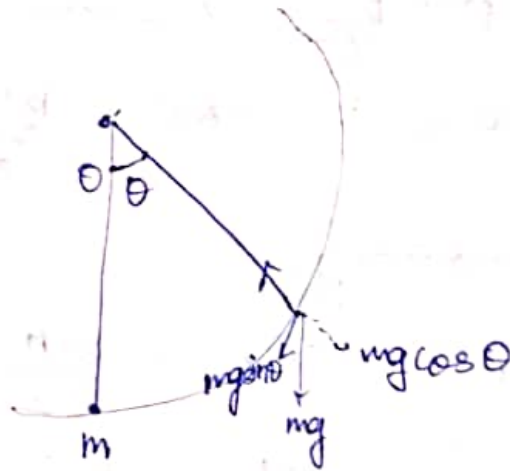
$$\frac{P}{\sin BOC} = \frac{Q}{\sin AOC} = \frac{R}{\sin BOA}$$

$$\frac{P}{\sin(90 + \frac{A}{2})} = \frac{Q}{\sin(90 + \frac{B}{2})} = \frac{R}{\sin(90 + \frac{C}{2})}$$

$$\frac{P}{\cos A/2} = \frac{Q}{\cos B/2} = \frac{R}{\cos C/2}$$

Q.5 (e)

One end of the inextensible string is fixed to a point O and to the other end is tied a particle of mass  $m$ . The particle is projected from its position of equilibrium vertically below O with horizontal velocity so as to carry it right round the circle. Prove that the sum of the tensions at the ends of a diameter is constant. (10)



$$T - mg \cos \theta = \frac{mv^2}{r}$$

$$mv \frac{dv}{dr} = -mg \sin \theta$$

$$mv \frac{dv}{r d\theta} = -mg \sin \theta$$

$$\frac{mv^2}{2r} = -mg(\cos \theta) + C_1$$

$$\text{at } \theta = \pi; v = 0$$

$$\Rightarrow C_1 = mg$$

$$\frac{mv^2}{r} = -2mg \cos \theta + 2mg$$

$$\Rightarrow T = mg \cos \theta + 2mg - 2mg \cos \theta$$

$$T = 2mg - mg \cos \theta$$

Candidates  
must not write  
on this margin

$$\begin{aligned} T(\theta) + T(\theta + \pi) &= 2mg - mg\cos\theta \\ &\quad + (2mg + mg\cos\theta) \\ &= 4mg = \text{constant} \end{aligned}$$

as mass is constant and  
 $g$  is constant.

Q.6 (a) Find the general solution of  $ayp^2 + (2x-b)p - y = 0, a > 0$ .

(15)

Candidates must not write on this margin

$$ayp^2 + (2x-b)p - y = 0$$

Solvable for  $p$ 

$$p = \frac{-(2x-b) \pm \sqrt{(2x-b)^2 + 4y(ay)}}{2ay}$$

Solvable for  $x$ 

$$(2x-b)p = y - ayp^2 + bp$$

$$x = \frac{y}{2p} - \frac{ayp}{2} + b$$

$$\frac{1}{p} \frac{dx}{dy} = \frac{1}{2p} + \frac{y}{2} \left( -\frac{1}{p^2} \right) \frac{dp}{dy} - \frac{ap}{2} - \frac{ay}{2} \frac{dp}{dy}$$

$$\frac{1}{2p} = \frac{dp}{dy} \left[ -\frac{y}{2p^2} - \frac{ay}{2} \right] - \frac{ap}{2}$$

$$\frac{ap^2 + 1}{2p} = -\frac{dp}{dy} \left[ \frac{ay + ay p^2}{2p^2} \right]$$

$$\frac{1 + ap^2}{2p} = -\frac{dp}{dy} \frac{y}{2p^2} (1 + ap^2)$$

$$\frac{(1+ap^2)}{2p} \left[ 1 + \frac{dp}{dy} \frac{y}{p} \right] = 0$$

Neglecting I factor as it leads to singular solution

$$\frac{dp}{dy} \frac{y}{p} + 1 = 0$$

$$\frac{dp}{p} = -\frac{dy}{y}$$

$$\log p = -\log y + \log c$$

$$p = \frac{c}{y}$$

Substituting in given equation

$$ay \left( \frac{c}{y} \right)^2 + (2x-b) \left( \frac{c}{y} \right) - y = 0$$

$$\frac{ac^2}{y} + (2x-b)c - y^2 = 0$$

is required general solution.

Find the law to force of the pole when path of a particle is the cardioid  $r = a(1 - \cos \theta)$  and prove that if  $F$  be the force at the apse and  $v$  the velocity there, then  $3v^2 = 4aF$ . (20)

$$\frac{P}{h^2 u^2} = \frac{du}{d\theta} + \left( \frac{du}{d\theta} \right)^2$$

$$P = \frac{1}{u^2} + \left( \frac{du}{d\theta} \right)^2$$

$$r = a(1 - \cos \theta)$$

$$\frac{1}{u} = a(1 - \cos \theta)$$

$$\frac{1}{u^2} \frac{du}{d\theta} = a \sin \theta$$

$$\frac{d}{d\theta} \left[ \frac{1}{u^3} \left( \frac{du}{d\theta} \right)^2 + \frac{1}{u^2} \frac{d^2 u}{d\theta^2} \right] = a \cos \theta$$

$$\frac{2u}{u^3} \left( -a \sin \theta \frac{du}{d\theta} \right)^2 + \frac{1}{u^2} \frac{d^2 u}{d\theta^2} = a \cos \theta$$

$$\frac{1}{u^2} \frac{d^2 u}{d\theta^2} = a \cos \theta + \frac{2u^2 a^2 \sin^2 \theta}{u^3}$$

$$\frac{d^2 u}{d\theta^2} = u^2 a \cos \theta + 2u a^2 \sin^2 \theta$$

$$P = \frac{1}{u^3} + \frac{d^2 u}{d\theta^2}$$

$$= \frac{1}{a^3 (1 - \cos \theta)^3} + \frac{2a^2 \sin^2 \theta}{a(1 - \cos \theta)} + \frac{2u^2 a^2 \sin^2 \theta}{u^3}$$

$$P = \frac{h^2 (a + 2a^2 \sin^2 \theta (1 - \cos \theta))}{a^4 (1 - \cos \theta)^4}$$

$$P = \frac{h^2}{a^5(1-\cos\theta)^5} (a^2(1-\cos\theta)^2)$$

$$P = h^2 \left[ \frac{1}{r^3} - \frac{a^2}{r^5} (\cos\theta - \cos^2\theta - 2\sin^2\theta) \right]$$

$$= h^2 \left[ \frac{1}{r^3} - \frac{a^2}{r^5} (\cos\theta - \cos^2\theta - 2(1-\cos^2\theta)) \right]$$

$$= h^2 \left[ \frac{1}{r^3} - \frac{a^2}{r^5} (\cos\theta - 2 + \cos^2\theta) \right]$$

$$h^2 \left[ \frac{1}{r^3} - \frac{a^2}{r^5} (\cos\theta(1-\cos\theta) - 2(1-\cos\theta)(1+\cos\theta)) \right]$$

$$= h^2 \left[ \frac{1}{r^3} - \frac{a^2}{r^5} ((1-\cos\theta)(-\cos\theta-2)) \right]$$

$$= h^2 \left[ \frac{1}{r^3} + \frac{(2+\cos\theta)a}{r^4} \right]$$

$$= h^2 \left[ \frac{1}{r^3} + \frac{2a+a}{r^4} - \frac{1}{r^3} \right]$$

$$P = \frac{h^2(3a)}{r^4}$$

At apse.  $\frac{du}{d\theta} = 0$

$$\Rightarrow \frac{du}{d\theta} = \frac{a \sin \theta}{a^2 (1 - \cos \theta)^2} = 0$$

$$\Rightarrow \theta = \pi$$

$$\Rightarrow \frac{1}{u} = 2a \Rightarrow u = \frac{1}{2a}$$

$$v^2 = h^2 \left[ u^2 \right] + \frac{du}{d\theta}$$

$$h^2 = \frac{1}{2} v^2$$

$$\Rightarrow F = \frac{4a^2 v^2 (3a)}{(2a)^4}$$

$$F = v^2 \left[ \frac{3}{4a} \right]$$

$$\Rightarrow 4aF = 3v^2$$

Evaluate  $\iint_S \text{curl } A \cdot d\mathbf{S}$ , where  $S$  is the open surface  $x^2 + y^2 - 4x + 4z = 0, z \geq 0$  and  $A = (y^2 + z^2 - x^2)\hat{i} + (2z^2 + x^2 - y^2)\hat{j} + (x^2 + y^2 - 3z^2)\hat{k}$ . (15)

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^2 - x^2 & 2z^2 + x^2 - y^2 & x^2 + y^2 - 3z^2 \end{vmatrix}$$

$$= \hat{i}(2y - 4z) + \hat{j}(2z - 2x) + \hat{k}(2x - 2y)$$

$$S: x^2 + y^2 - 4x + 4z = 0$$

$$\nabla S = (2x - 4)\hat{i} + 2y\hat{j} + 4\hat{k}$$

$$(\nabla \times A) \cdot \nabla S = (2y - 4z)(2x - 4) + 2y(2z - 2x) + 4(2x - 2y)$$

$$x^2 + y^2 - 4x = 0 \Rightarrow (x-2)^2 + y^2 = 2^2$$

$$\int A \cdot d\mathbf{r} = \int (y^2 - x^2)dx + (x^2 - y^2)dy$$

Using Stokes theorem ~~is~~

$$x = 2 \cos \theta + 2$$

$$y = 2 \sin \theta$$

$$dx = -2 \sin \theta d\theta$$

$$dy = 2 \cos \theta d\theta$$

$$= \int_0^{2\pi} (x^2 - y^2)(2 \sin \theta + 2 \cos \theta) d\theta$$

$$= \int_0^{2\pi} (4 + 4 \cos^2 \theta + 8 \cos \theta - 4 \sin^2 \theta)(\sin \theta + \cos \theta) d\theta$$

$$= 16\pi$$

(a) Solve  $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$ .

(20)

~~$y = e^x$~~

~~$$[(x+2) - (2x+5) + 2]e^x = -(x+1)e^x$$~~

~~$y = e^{2x}$~~

~~$$[4(x+2) - 2(2x+5) + 2]e^{2x}$$~~

~~$y = x^2$~~

~~$$2(x+2) - 2x(2x+5) + 2x^2$$~~

~~$y = e^{2x}$~~

~~$$[4(x+2) - 2(2x+5) + 2]e^{2x}$$~~

~~$$= 0$$~~

$\Rightarrow y = e^{2x}$  is part of solution of homogeneous.

let  $y = uv$  is general solution then.

$$\frac{d^2v}{dx^2} + \left( p + \frac{2}{u} \frac{du}{dx} \right) \frac{dv}{dx} = \frac{R(x)}{u}$$

$$\frac{d^2 y}{dx^2} - \frac{(2x+5)}{x+2} \frac{dy}{dx} + \frac{2y}{x+2} = \frac{x+1}{x+2} e^x$$

$$y_2 + P y_1 + Q y = R$$

$$P = -\frac{(2x+5)}{x+2}; \quad Q = \frac{2}{x+2};$$

$$R = \frac{x+1}{x+2} e^x.$$

Substitution

$$\frac{d^2 v}{dx^2} + \left( -\frac{(2x+5)}{x+2} + \frac{2}{e^{2x}} 2e^{2x} \right) \frac{dv}{dx} = \frac{x+1}{x+2} \frac{e^x}{e^{2x}}$$

$$\frac{d^2 v}{dx^2} + \left( \frac{2x+3}{x+2} \right) \frac{dv}{dx} = \frac{x+1}{x+2} e^{-x}$$

$$\text{let } \frac{dv}{dx} = w$$

$$\frac{dw}{dx} + \left( \frac{2x+3}{x+2} \right) w = \frac{x+1}{x+2} e^{-x}$$

It is linear ODE.

$$\begin{aligned} \text{IF} &= \int e^{+\left(\frac{2x+3}{x+2}\right) dx} \\ &= \int e^{+\left(\frac{x}{2} - \frac{1}{x+2}\right) dx} \\ &= \int e^{\frac{x}{2}} \cdot e^{-\frac{1}{x+2}} dx \\ &= \frac{e^{+2x}}{x+2} \end{aligned}$$

$$w \left( \frac{e^{-2x}}{x+2} \right) = \int \frac{x+1}{x+2} e^{-x} \frac{e^{+2x}}{x+2} dx$$

$$\begin{aligned}
 w\left(\frac{e^{-2x}}{x+2}\right) &= \int e^x \frac{x+1}{(x+2)^2} dx = \int e^x \left[ \frac{x+2-1}{(x+2)^2} \right] dx \\
 &= \int e^x \left[ \frac{1}{x+2} - \frac{1}{(x+2)^2} \right] dx \\
 &= \frac{e^x}{x+2} + c_1 \left( \int e^x (f(x) + f'(x)) \right) \\
 &= e^x f(x)
 \end{aligned}$$

$$\Rightarrow w = e^{3x} + (x+2)e^{2x} c_1$$

$$\frac{dw}{dx} = e^{3x} + e^{2x} (x+2) c_1$$

$$v = \frac{e^{3x}}{3} + c_1 \left[ \frac{e^{2x} x}{2} - \frac{e^{2x}}{4} + e^{2x} \right] + c_2$$

$$y = uv$$

$$= e^{2x} v$$

$$= \frac{e^{5x}}{3} + c_1 \left[ \frac{x e^{4x}}{2} + \frac{3}{4} e^{4x} \right] + c_2 e^{2x}$$

Q.7 (b)

Suppose  $A = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ . Evaluate  $\int_C A \cdot d\mathbf{r}$  from  $(0,0,0)$  to  $(1,1,1)$  along the following paths C:

(i)  $x = t, y = t^2, z = t^3$ .

(ii) The straight lines from  $(0,0,0)$  to  $(1,0,0)$ , then to  $(1,1,0)$  and then to  $(1,1,1)$

(iii) The straight line joining  $(0,0,0)$  and  $(1,1,1)$ . (15)

$$i) \int A \cdot d\mathbf{r} = \int (3x^2 + 6y) dx - 14yz dy + 20xz^2 dz$$

$$x = t; y = t^2; z = t^3$$

$$dx = dt; dy = 2t dt; dz = 3t^2 dt$$

$$\int_{t=0}^1 (3t^2 + 6t^2) dt - 14(t^2)(t^3)(2t dt) + 20(t)(t^3)^2(3t^2 dt)$$

$$= \int_0^1 9t^2 - 28t^6 + 60t^9 dt$$

$$= 3 - 4 + 6 = \boxed{5}$$

ii)

$$(0,0,0) \rightarrow (1,0,0)$$

$$dy = dz = y = z = 0$$

$$A \cdot d\mathbf{r} = \int_0^1 3x^2 dx = \boxed{1}$$

$$(1, 0, 0) \rightarrow (1, 1, 0)$$

$$x = 1; \quad dx = dz = dy = 0$$

$$\int A \cdot dx = \boxed{0}$$

$$(1, 1, 0) \rightarrow (1, 1, 1)$$

$$x = y = 1; \quad dx = dy = 0$$

$$\int A \cdot dz = \int_0^1 20xz^2 dz = \int_0^1 20z^2 dz$$

$$= \boxed{\frac{20}{3}}$$

$$\text{Ans} \quad 1 + 0 + \frac{20}{3} = \boxed{\frac{23}{3}}$$

iii)  $x = y = z = t$   
 $dx = dy = dz$

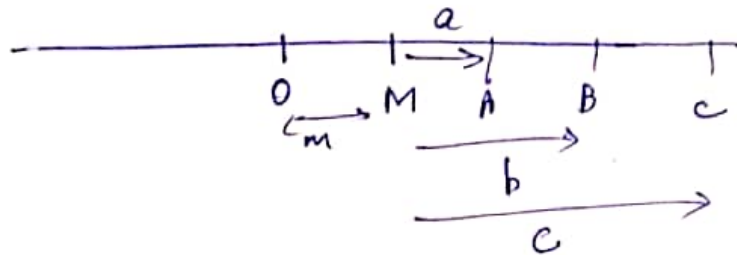
$$\int A \cdot dr = \int_0^1 (3t^2 + 6t - 14t^2 + 20t^3) dt$$

$$= 1 + 3 - \frac{14}{3} + 5$$

$$= \boxed{\frac{13}{3}}$$

- (c) If in a simple harmonic motion  $u, v, w$  be the velocities at a distance  $a, b, c$  from a fixed point on the straight line (which is not the centre of force), show that the period  $T$  is given

by the equation  $\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$ . (15)



$$\begin{aligned}
 u &= \omega \sqrt{a^2 - (m+a)^2} \\
 v &= \omega \sqrt{a^2 - (m+b)^2} \\
 w &= \omega \sqrt{a^2 - (m+c)^2}
 \end{aligned}$$

Now  $\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \Sigma u^2(b-c)$

$$\begin{aligned}
 &= u^2(b-c) + v^2(c-a) + w^2(a-b) \\
 &= \omega^2 [-(m+a)^2(b-c) - (m+b)^2(c-a) - (m+c)^2(a-b)] \\
 &= \omega^2 [a^2(c-b) + b^2(a-c) + c^2(b-a) \\
 &\quad a^2(c-b) + a(b^2-c^2) + bc(c-b)] \\
 &= \frac{4\pi^2}{T^2} [c-b] [a^2 - ac + bc - ab] \quad \begin{matrix} ab - ac \\ + bc - ba \\ ca - cb \end{matrix} \\
 &= \frac{4\pi^2}{T^2} (b-c)(c-a)(a-b)
 \end{aligned}$$

$$LHS = RHS$$