

# NEXT IAS

T-5/43

(To be filled by candidate)

Name of Candidate : ANNU KOUSIK NARASIMHA

Roll No. : DACMTTA003 Contact No. : 64414 284

e-mail : K Date of Examination : 21-07-2024

Exam Centre : Old Rajinder Nagar ☒ Online ☐

## CSE (MAINS) TEST SERIES-2024

Test-05

MATHEMATICS  
OPTIONAL

Dated : 21-07-2024

Time Allowed : Three Hours

Maximum Marks: 250

### QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are **EIGHT** questions divided in **Two** sections and printed in **ENGLISH**.

Candidate has to attempt **FIVE** questions in all.

**Question No. 1 and 5** are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

#### DELHI CENTRE:

**Old Rajinder Nagar:** 27-B, Pusa Road, Metro Pillar no. 118, Near Karol Bagh Metro, New Delhi-110060 | Phone: 8081300200

**Mukherjee Nagar:** 1422, Main Mukherjee Nagar Road, Near Batra Cinema, New Delhi-110009 | Phone: 8081300200

**BHOPAL CENTRE:** Plot No. 46 Zone - 2, M.P Nagar, Bhopal - 462011 | Phone: 8827664612, 8081300200

**JAIPUR CENTRE:** Plot No. 6 & 7, 3rd Floor, Sree Gopal Nagar, Gopalpura Bypass, Jaipur - 302015 | Phone: 9358200511

**PRAYAGRAJ CENTRE:** 31/31 Sardar Patel Marg, Civil Lines, Prayagraj, Uttar Pradesh - 211001 | Phone: 9958857757

(For filling by Examiners only)

| Q. No.      | Page No. | Max. Marks | Marks | Total | Signature |
|-------------|----------|------------|-------|-------|-----------|
| 1. (a)      | 4        | 10         |       |       |           |
| 1. (b)      | 6        | 10         |       |       |           |
| 1. (c)      | 8        | 10         |       |       |           |
| 1. (d)      | 10       | 10         |       |       |           |
| 1. (e)      | 12       | 10         |       |       |           |
| 2. (a)      | 14       | 20         |       |       |           |
| 2. (b)      | 18       | 15         |       |       |           |
| 2. (c)      | 21       | 15         |       |       |           |
| 3. (a)      | 24       | 15         |       |       |           |
| 3. (b)      | 27       | 15         |       |       |           |
| 3. (c)      | 30       | 20         |       |       |           |
| 4. (a)      | 34       | 20         |       |       |           |
| 4. (b)      | 38       | 15         |       |       |           |
| 4. (c)      | 41       | 15         |       |       |           |
| 5. (a)      | 44       | 10         |       |       |           |
| 5. (b)      | 46       | 10         |       |       |           |
| 5. (c)      | 48       | 10         |       |       |           |
| 5. (d)      | 50       | 10         |       |       |           |
| 5. (e)      | 52       | 10         |       |       |           |
| 6. (a)      | 54       | 15         |       |       |           |
| 6. (b)      | 57       | 20         |       |       |           |
| 6. (c)      | 61       | 15         |       |       |           |
| 7. (a)      | 64       | 20         |       |       |           |
| 7. (b)      | 68       | 15         |       |       |           |
| 7. (c)      | 71       | 15         |       |       |           |
| 8. (a)      | 74       | 20         |       |       |           |
| 8. (b)      | 78       | 15         |       |       |           |
| 8. (c)      | 81       | 15         |       |       |           |
| Grand Total |          |            |       |       |           |

Remarks

**Observations:**

## SECTION-A

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- Q.1 (a) Let  $U$  and  $W$  be subspaces of vector space  $V$  and  $x, y \in V$ . Then prove that  $x+U \subseteq y+W$  iff  $U \subseteq W$  and  $x-y \in W$ . (10)

I Consider  $U \subseteq W$  &  $x, y \in W$

$$x-y \in W \Rightarrow x, y \in W \quad (\because W \text{ is a subspace, } x, y \in W \Rightarrow x+y \in W)$$

Considering ~~the~~ the subspace  $U$ ,

$$U \subseteq W \Rightarrow x+U \subseteq W \quad \text{--- (1)} \quad (\because x \in W)$$

$$\text{Also } W = y+W \quad (\because y \in W) \quad \text{--- (2)}$$

$$\therefore x+U \subseteq y+W \quad (\text{From (1) \& (2)})$$

$$\Rightarrow U \subseteq W \text{ \& } x-y \in W \Rightarrow x+U \subseteq y+W$$

II Let  $x+U \subseteq y+W$

$x+U$  is a subspace with elements

$$x+u_1, x+u_2, \dots \text{ where } u_1, u_2, \dots \in U$$

~~We use the fact that one subspace is a subspace subset of other subspace iff one is contained in the other~~

Let  $x + u_p$  be an element of  $x + U$

$$x + u_p = y + w_q \quad (\because u \subseteq w)$$



Q.1 (b)

Show that 0 is characteristic root of matrix iff the matrix is singular.

(10)

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① Let 0 be a characteristic root

$$\Rightarrow |A - 0I| = 0 \quad (\because |A - \lambda I| = 0 \text{ for char root } \lambda)$$

$$\Rightarrow |A| = 0$$

$\Rightarrow A$  is singular. — ①.

② Let  $A$  is a singular matrix

$$\Rightarrow |A| = 0 \text{ — ② let } \lambda \text{ be a char root of } A.$$

$$\Rightarrow AX = \lambda X$$

$$\Rightarrow (A - \lambda I)X = 0I. \text{ Taking modulus on both sides:}$$

$$|(A - \lambda I)X| = |0I| = 0$$

$$\Rightarrow |A - \lambda I| |X| = 0 \quad (\because |AB| = |A| |B|)$$

$$\Rightarrow \text{Since } |X| \neq 0,$$

$$|A - \lambda I| = 0 \text{ — ③}$$

② and ③ hold good when

$$\lambda = 0 \Rightarrow '0' \text{ is a char root.}$$

$\therefore A$  is singular  $\Leftrightarrow 0$  is char root of  $A$ .



Q.1 (c)

Evaluate  $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}$  as  $x \rightarrow 0$ .

(10)

Consider  $l = (\cos x)^{\frac{1}{x^2}}$ ;  $\lim_{x \rightarrow 0} l = \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}$

$$\ln l = \frac{1}{x^2} \ln(\cos x)$$

$$\lim_{x \rightarrow 0} \ln l = \lim_{x \rightarrow 0} \frac{1}{x^2} \ln(\cos x)$$

Applying expansion,  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$$\lim_{x \rightarrow 0} \ln l = \lim_{x \rightarrow 0} \frac{1}{x^2} \ln \left( 1 + \left( -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right) \right)$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots$$

$$\Rightarrow \lim_{x \rightarrow 0} \ln l = \lim_{x \rightarrow 0} \frac{1}{x^2} \left[ \left( -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right) - \frac{1}{2} \left( \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right)^2 + \dots \right]$$

$$= \lim_{x \rightarrow 0} \left( -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots \right)$$

$$= -\frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow 0} \ln l = -\frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow 0} l = e^{-1/2}$$

$$\Rightarrow \lim_{x \rightarrow 0} (\cos x)^{1/x^2} = e^{-1/2}$$

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Q.1 (d)

Test the convergence of  $\int_0^1 \frac{\sin\left(\frac{1}{x}\right)}{\sqrt{x}} dx$ 

(10)

By comparision test, we know  
that if  $\int_a^b f(x) \leq \int_a^b g(x) dx$  and  
 $\int_a^b g(x) dx$  is convg, then  $\int_a^b f(x) dx$  is also  
convg

Considering  $\int f(x) = \int \frac{\sin(1/x)}{\sqrt{x}} dx$ .

$$\Rightarrow \int |f(x)| = \int \left| \frac{\sin(1/x)}{\sqrt{x}} \right| dx \leq \int \frac{1}{\sqrt{x}} dx$$

$$(\because |\sin(1/x)| \in [0, 1])$$

$$\text{let } \int g(x) dx = \int \frac{1}{\sqrt{x}} dx$$

We know that  $\int_a^b \frac{1}{(x-a)^n} dx$  is convg if  $n < 1$

$$\Rightarrow \int \frac{1}{\sqrt{x}} dx = \int \frac{1}{(x-0)^{1/2}} dx \text{ is convergent}$$

( $\because 1/2 < 1$ )

$\therefore \int g(x) dx$  is convergent

$\therefore$  By comparison test,  $\int |f(x)| dx$  is convg

Now,  $\int_0^1 \left| \frac{\sin^{-1} x}{\sqrt{x}} \right| dx$  is absolutely convergent

$\therefore \int_0^1 \frac{\sin^{-1} x}{\sqrt{x}} dx$  is convergent

( $\because$  absolute convergence implies regular convergence)



- Q.1 (e) Find the equation of the cylinder whose generators touch the sphere  $x^2 + y^2 + z^2 = 9$  and are perpendicular to the plane  $x - y - 3z = 5$ . (10)

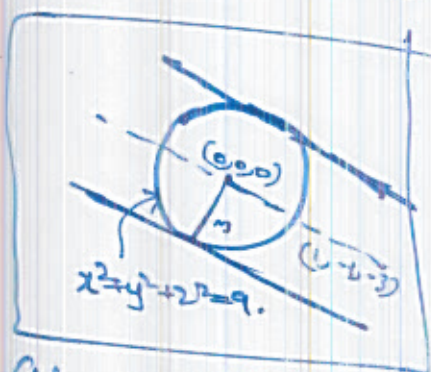
The dir of generators are  $\perp$  to the normal of the plane  $x - y - 3z = 5$ .

Radius of sphere = 3 units =  $r$ .

Since the cylinder touch the sphere, the centre of sphere lies on axis of the cylinder  $\rightarrow (0, 0, 0)$  lies on axis  $= (x, y, z) (1 \text{ say})$

The dir of the axis of cylinder  $= (1, -1, -3)$   
 $= (l, m, n) \text{ (say)}$

Now the eqn of cylinder with given conditions above is:



$$\frac{(l(x-\alpha) + m(y-\beta) + n(z-\gamma))^2}{(l^2 + m^2 + n^2)} = \frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - r^2}{1}$$

$$\Rightarrow (1(x-0) + (-1)(y-0) + (-3)(z-0))^2 = (1^2 + (-1)^2 + (-3)^2)$$

$$(x-0)^2 + (y-0)^2 + (z-0)^2 - 3^2$$

$$\Rightarrow (x - y - 3z)^2 = 11 (x^2 + y^2 + z^2 - 9)$$

$$\Rightarrow x^2 + y^2 + 9z^2 - 2xy + 6yz - 6xz = 11x^2 + 11y^2 + 11z^2 - 99$$

$$\Rightarrow 10x^2 + 10y^2 + 2z^2 + 2xy - 6yz + 6xz - 99 = 0$$

is the read can of cylinder.



Q.2 (a) Find non-singular matrices  $P, Q$  so that  $PAQ$  is a normal form where

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \text{ and hence find its rank.} \quad (20)$$

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- Q.2 (b) Find the whole area included between the curve  $x^2y^2 = a^2(y^2 - x^2)$  and its asymptotes. (15)

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Q.2 (c)

Tangent planes are drawn to the ellipsoid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  through the point  $(\alpha, \beta, \gamma)$ .

Prove that the perpendiculars to them through the origin generate the cone

$$(\alpha x + \beta y + \gamma z)^2 = a^2 x^2 + b^2 y^2 + c^2 z^2. \quad (15)$$



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Q.3 (a)

For what values of  $\lambda$  and  $\mu$ , the system of equations

$$\begin{aligned} x+y+z &= 6 \\ x+2y+3z &= 10 \\ x+2y+\lambda z &= \mu \end{aligned} \text{ has}$$

- (i) no solution
- (ii) a unique solution
- (iii) an infinite number of solutions

(15)

Writing the system of equations in the form  $AX = B$ :

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$A \quad X = B$

Writing the augmented Matrix  $[A|B]$  and performing row operations:

$$[A|B] = \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array} \right] \xrightarrow{\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1}} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array} \right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array} \right] \text{ is the reqd form.}$$

(i) No solution - The system has no solution if  $\text{Rank}(A) \neq \text{Rank}(A|B)$

Taking  $\text{Rank}(A) = 2$  and  $\text{Rank}(A/B) = 3$ ,  
we get no solution.

$\therefore \lambda = 3, \mu \neq 10$  is the condition for  
no solution  
( $\because \text{Rank} = \text{No. of non-zero rows}$ )

(ii) Unique solution - The condition is

$\text{Rank}(A) = \text{Rank}(A/B) = n$  where  $n$  is the  
no. of variables

$$\Rightarrow \text{Rank}(A) = \text{Rank}(A/B) = 3$$

$\Rightarrow \lambda \neq 3, \mu \neq 10$  is the condition for uni-  
-que solution.

(iii) Infinite no. of solutions - The condition is

$$\text{Rank}(A) \neq \text{Rank}(A/B) < n$$

$$\Rightarrow \text{Rank}(A) = \text{Rank}(A/B) < 2$$

$\Rightarrow \lambda = 3, \mu = 10$  is the reqd condition  
for infinite solutions

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Q.3 (b)

Find the shortest distance between the line  $y = 10 - 2x$  and the ellipse  $\frac{x^2}{4} + \frac{y^2}{9} = 1$  using Lagrange's method of multipliers. (15)

Let  $(x, y)$  be the reqd point on the ellipse. Shortest distance from the line  $2x + y - 10 = 0$  is  $\frac{|2x + y - 10|}{\sqrt{2^2 + 1^2}} = \frac{|2x + y - 10|}{\sqrt{5}}$

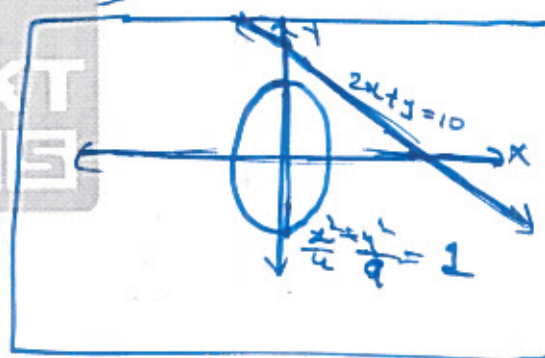
( $\because$  shortest distance b/w  $(x, y)$  &  $ax + by + c = 0$

$$\text{is } \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

To find:

$$\min \frac{|2x + y - 10|}{\sqrt{5}} \text{ subject to}$$

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0.$$



Let  $f(x, y) = 10 - 2x - y$  and  $\phi(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1$   
 ( $\because$  origin lies on same side of shortest distance)

$$\# d(f - \lambda \phi) = 0 \quad (\lambda \text{ is Lagrange multiplier})$$

$$\Rightarrow \left( \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} \right) dx + \left( \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} \right) dy = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\Rightarrow (-2) + \lambda \left( \frac{2x}{4} \right) = 0 \quad \text{and} \quad (-1) + \lambda \left( \frac{2y}{9} \right) = 0$$

$$\Rightarrow \boxed{x = 4/\lambda \quad \text{and} \quad y = 9/2\lambda}$$

From the eqn of ellipse,

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\Rightarrow \frac{(4/\lambda)^2}{4} + \frac{(9/\lambda)^2}{9} = 1$$

$$\Rightarrow \frac{16}{4\lambda^2} + \frac{81}{4\lambda^2 \times 9} = 1$$

$$\Rightarrow \frac{4}{\lambda^2} + \frac{9}{4\lambda^2} = 1$$

$$\Rightarrow 4\lambda^2 = 25 \Rightarrow \boxed{\lambda = \pm \frac{5}{2}}$$

Taking  $\lambda = 5/2$ , we get

$$(x, y) = \left(\frac{8}{5}, \frac{9}{5}\right)$$

$$\Rightarrow d = \frac{|2x + y - 10|}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \underline{\underline{\sqrt{5} \text{ units}}}$$

Taking  $\lambda = -5/2$ , we get

$$(x, y) = \left(-\frac{8}{5}, -\frac{9}{5}\right)$$

$$\Rightarrow d = \frac{|2x + y - 10|}{\sqrt{5}} = \underline{\underline{3\sqrt{5} \text{ units}}}$$

$\therefore$  shortest distance b/w the line and ellipse is  $\sqrt{5}$  units

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- Q.3 (c) Prove that  $5x^2 + 5y^2 + 8z^2 + 8yz + 8zx - 2xy + 12x - 12y + 6 = 0$  represents a cylinder whose cross section is an ellipse of eccentricity  $1/\sqrt{2}$ . (20)

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- Q.4 (a) Show that the set  $\{(1, 2, 0, 3), (4, 0, 5, 8), (8, 1, 5, 6)\}$  is linearly independent in  $\mathbb{R}^4$ . The vectors form a basis for a three-dimensional subspace  $V$  of  $\mathbb{R}^4$ . Construct an orthonormal basis for  $V$ . (20)

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Q.4 (b) Find the asymptotes of  $x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0$ .

(15)

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Q.4 (c)

Prove that the centres of the spheres which touch the lines  $y = mx, z = c$  and  $y = -mx, z = -c$  must lie on the conicoid  $mx + c(1 + m^2)z = 0$ . (15)

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## SECTION-B

- Q.5 (a) Find the Wronskian of the set of functions  $\{3x^3, |x^3|\}$  on the interval  $[-1, 1]$  and determine whether the set is linearly dependent on  $[-1, 1]$ . (10)

$$\text{Let } f(x) = 3x^3 \text{ and } g(x) = |x^3|$$

$$g(x) = \begin{cases} -x^3, & x \in [-1, 0] \\ x^3, & x \in [0, 1] \end{cases}$$

$$\text{Now wronskian} = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} :$$

For the interval  $[-1, 0]$ :

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} 3x^3 & -x^3 \\ 9x^2 & -3x^2 \end{vmatrix} = -3x^5 + 3x^5 = 0$$

For the interval  $[0, 1]$ :

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} 3x^3 & x^3 \\ 9x^2 & 3x^2 \end{vmatrix} = 3x^5 - 3x^5 = 0$$

$$\boxed{\therefore \text{Wronskian} = 0 \quad \forall x \in [-1, 1]}$$

The set is linearly dependent/ independent based on any relation b/w  $f(x)$  and  $g(x)$

$$\text{Here } f(x) = 3x^3$$

$$g(x) = |x^3|$$

For  $[-1, 0]$ ,  $f(x) = -3g(x)$

For  $[0, 1]$ ,  $f(x) = 3g(x)$

Hence, there exists no direct relation  
b/w  $f(x)$  &  $g(x)$ .

$\therefore$  The set is linearly Independent



Q.5 (b) The necessary and sufficient condition for  $f(t)$  to be constant is  $\frac{df}{dt} = 0$ .

(10)

Necessary Condition

$$\text{Let } \vec{f}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

where  $x, y, z$  are constants.

$$\begin{aligned} \Rightarrow \vec{f}(t) &= \frac{d}{dt}(x)\vec{i} + \frac{d}{dt}(y)\vec{j} + \frac{d}{dt}(z)\vec{k} \\ &= 0\vec{i} + 0\vec{j} + 0\vec{k} = \underline{\underline{0}}. \end{aligned}$$

$\therefore$  If  $f(t) = \text{constant}$ , then  $\frac{df}{dt} = 0$

Sufficient condition

$$\text{Let } \frac{d\vec{f}}{dt} = 0 \Rightarrow \vec{f} \cdot \frac{d\vec{f}}{dt} = \vec{f} \cdot \vec{0} = 0$$

(Multiplying  $\vec{f}$  on both sides by dot product)

$$\Rightarrow \vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \Rightarrow 2\left(\vec{f} \cdot \frac{d\vec{f}}{dt}\right) = 0$$

$$\Rightarrow \vec{f} \cdot \frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt} \cdot \vec{f} = 0 \quad (\because \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A})$$

$$\Rightarrow \frac{d}{dt}(\vec{f} \cdot \vec{f}) = 0$$

$$\Rightarrow |\vec{f}|^2 = c \Rightarrow |\vec{f}| = k$$

$\therefore |f| = C$  where  $C$  is any constant.

$\therefore f$  is a constant vector for  $\frac{df}{dt} = 0$

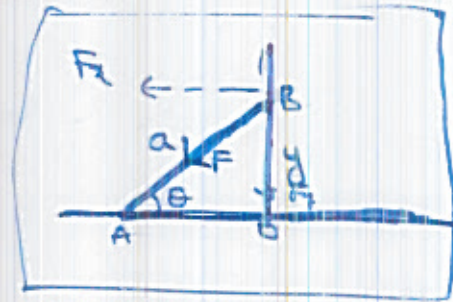


Q.5 (c)

At what point of a tree must one end of a rope of given length 'a' be attached so that a man pulling at the other end may exert greatest force to pull it over? (10)

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Let the rope be attached at a height  $y'$  above the ground



From the figure  $\sin \theta = y/a$

The force can be resolved as  $F_x = F \cos \theta$ ;  $F_y = F \sin \theta$

$$\vec{F} = F_x \vec{i} + F_y \vec{j}$$

$$\vec{F} = F \cos \theta \vec{i} + F \sin \theta \vec{j}$$

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Q.5 (d)

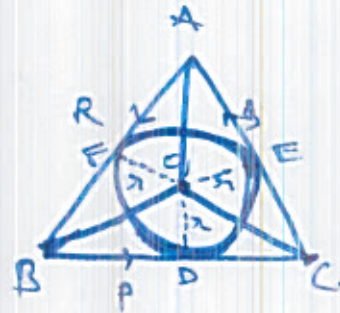
A particle is placed at the centre of a circle and is acted on by three forces  $P, Q$  and  $R$  tending towards the angular points  $A, B$  and  $C$  of a circumscribed triangle. Prove that when

the forces are in equilibrium then  $\frac{P}{\cos \frac{A}{2}} = \frac{Q}{\cos \frac{B}{2}} = \frac{R}{\cos \frac{C}{2}}$ . (10)

The resultant of the  
forces passes through the  
~~centre~~<sup>in</sup> centre of the  
triangle

$$\Rightarrow P\vec{r} + Q\vec{r} + R\vec{r} = \vec{0}$$

$$\Rightarrow P + Q + R = 0$$



$$\angle BOD = \frac{\pi}{2} - \frac{B}{2}$$

$$\angle COE = \frac{\pi}{2} - \frac{C}{2}$$

$$\angle BOE = \frac{\pi}{2} + \frac{A}{2}$$

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Q.5 (e)

One end of the inextensible string is fixed to a point O and to the other end is tied a particle of mass  $m$ . The particle is projected from its position of equilibrium vertically below O with horizontal velocity so as to carry it right round the circle. Prove that the sum of the tensions at the ends of a diameter is constant. (10)

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The projection velocity is enough to

carry around the

circle  $\Rightarrow T \geq 0$  at all points

The Tension is given by

$$T = \frac{m}{a} (u^2 - 2ag + 3ag \cos \theta)$$

Taking  $T = 0$  at highest point,

$$0 = \frac{m}{a} (u^2 - 2ag + 3ag \cos(\pi)) \Rightarrow \boxed{u^2 = 5ag}$$

~~Let~~

Let  $AA'$  be any diameter with

$$\angle POA = \theta \Rightarrow \angle POA' = \pi + \theta$$

$$\text{Tension at A} = \frac{m}{a} (u^2 - 2ag + 3ag \cos \theta)$$

$$\Rightarrow T_A = \frac{m}{a} (5ag - 2ag + 3ag \cos \theta)$$

$$T_A = \frac{m}{a} (3ag + 3ag \cos \theta) \quad \text{--- (1)}$$

$$\text{At A'}, T_{A'} = \frac{m}{a} (u^2 - 2ag + 3ag \cos(\theta + \pi))$$

$$\Rightarrow T_{A'} = \frac{m}{a} (5ag - 2ag + 3ag \cos \theta)$$

$$\Rightarrow T_{A'} = \frac{m}{a} (3ag - 3ag \cos \theta) \quad (\because \cos(\theta + \pi) = -\cos \theta)$$

$$\Rightarrow T_A + T_{A'} = \frac{m}{a} (3ag + 3ag \cos \theta) + \frac{m}{a} (3ag - 3ag \cos \theta)$$

$$\Rightarrow T_A + T_{A'} = \frac{m}{a} (6ag) = \underline{\underline{6mg}}$$

$$\therefore \underline{T_A + T_{A'} = 6mg = \text{constant}},$$

since  $m$  is constant

Hence proved.

Q.6 (a) Find the general solution of  $ayp^2 + (2x-b)p - y = 0, a > 0$ .

(15)

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~~Soluble for p form:~~

~~$ayp^2 + (2x-b)p - y = 0$~~

~~$\Rightarrow p = \frac{-(2x-b) \pm \sqrt{(2x-b)^2 - 4(ay)(-y)}}{2ay}$~~

~~$\Rightarrow p = \frac{(b-2x) \pm \sqrt{4x^2 + b^2 - 4bx + 4ay^2}}{2ay}$~~

$ayp^2 + (2x-b)p - y = 0$

$\Rightarrow (2x-b)p = y - ayp^2 \Rightarrow 2x \cdot b = \frac{y}{p} - ayp$

$\Rightarrow 2x = b + \frac{y}{p} - ayp$

$\Rightarrow x = \frac{b}{2} + \frac{y}{2p} - \frac{ayp}{2}$  . Diff w.r.t y on both sides

$\frac{dx}{dy} = \frac{1}{p} = \frac{d}{dy} \left( \frac{y}{2p} \right) - \frac{d}{dy} \left( \frac{ayp}{2} \right)$

$\Rightarrow \frac{1}{p} = \frac{1}{2p} + \left( \frac{y}{2} \right) \left( -\frac{1}{p^2} \frac{dp}{dy} \right) - \frac{a}{2} \left[ p + y \frac{dp}{dy} \right]$

$\Rightarrow \frac{1}{2p} = -\frac{y}{2p^2} \frac{dp}{dy} - \frac{ap}{2} - \frac{ay}{2} \frac{dp}{dy}$

$\Rightarrow \frac{1}{2p} + \frac{ap}{2} = \frac{dp}{dy} \left( \frac{y}{2p^2} - \frac{ay}{2} \right)$

$\Rightarrow \frac{1+ap^2}{2p} = \left( \frac{dp}{dy} \right) y \left( \frac{1}{2p^2} - \frac{a}{2} \right)$

$$\Rightarrow \frac{1+ap^2}{2p} = \left( \frac{dp}{dy} \right) (y) \left( \frac{1-ap^2}{2p^2} \right)$$

$$\Rightarrow \frac{(1+ap^2)}{(1-ap^2)} p = y \frac{dp}{dy}$$

$$\Rightarrow \frac{dy}{y} = \left( \frac{1-ap^2}{1+ap^2} \right) \times \frac{1}{p} dp \quad \text{--- (1)}$$

$$\text{Let } ap^2 = t \Rightarrow 2ap dp = dt$$

$$\Rightarrow p dp = \frac{1}{2a} dt.$$

$$\frac{1}{p} dp = \frac{p dp}{p^2} = \frac{1}{2a} dt \times \frac{a}{t} = \frac{dt}{2t}$$

① becomes

$$\Rightarrow \frac{dy}{y} = \frac{1-t}{1+t} \times \frac{dt}{2t} = \frac{1}{2} \left( \frac{1-t}{t^2+t} \right) dt.$$

Integrating,

$$\int \frac{dy}{y} = \frac{1}{2} \int \left( \frac{1}{t^2+t} \right) - \left( \frac{1}{t+1} \right) dt.$$

$$= \frac{1}{2} \int \frac{1}{t} - \frac{2}{t+1} dt.$$

$$\Rightarrow \ln y = \frac{1}{2} [\ln t - 2 \ln (t+1)]$$

$$\Rightarrow \ln y^2 = \ln \left( \frac{t}{(t+1)^2} \right) \Rightarrow y^2 = K \left( \frac{ap^2}{(1+ap^2)^2} \right)$$

and  $ayp^2 + (2x-b)p - y = 0$  together give the solution.

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Q.6 (b)

Find the law to force of the pole when path of a particle is the cardioid  $r = a(1 - \cos \theta)$  and prove that if  $F$  be the force at the apse and  $v$  the velocity there, then  $3v^2 = 4aF$ . (20)

$$\text{Given that } r = a(1 - \cos \theta)$$

$$\Rightarrow r = a\left(2 \sin^2 \frac{\theta}{2}\right) \quad \left(\because 1 - \cos 2A = 2 \sin^2 A\right)$$

$$\Rightarrow u = \frac{1}{2a} \operatorname{cosec}^2 \frac{\theta}{2} \quad (\because u = 1/r)$$

$$\begin{aligned} \Rightarrow \frac{du}{d\theta} &= \frac{1}{2a} \frac{d}{d\theta} \left( \operatorname{cosec}^2 \frac{\theta}{2} \right) \\ &= \frac{1}{2a} \times 2 \operatorname{cosec} \frac{\theta}{2} \times \left( -\operatorname{cosec} \frac{\theta}{2} \cot \frac{\theta}{2} \right) \times \frac{1}{2} \\ &= -\frac{1}{2a} \operatorname{cosec}^2 \frac{\theta}{2} \cot \frac{\theta}{2} \quad \left(\because \frac{d}{d\theta} (\operatorname{cosec} A) = -\operatorname{cosec} A \cot A\right) \end{aligned}$$

$$\Rightarrow \frac{du}{d\theta} = -u \cot \frac{\theta}{2}$$

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta} \left( -u \cot \frac{\theta}{2} \right)$$

$$= - \left[ \frac{du}{d\theta} \cot \frac{\theta}{2} + u \frac{d}{d\theta} \left( \cot \frac{\theta}{2} \right) \right]$$

$$= - \left[ -\frac{1}{2a} \operatorname{cosec}^2 \frac{\theta}{2} \cot^2 \frac{\theta}{2} + \frac{u}{2} \times -\operatorname{cosec}^2 \frac{\theta}{2} \right]$$

$$= - \left[ -u \cot^2 \frac{\theta}{2} - \frac{u}{2} \operatorname{cosec}^2 \frac{\theta}{2} \right]$$

$$= u \cot^2 \frac{\theta}{2} + \frac{u}{2} \operatorname{cosec}^2 \frac{\theta}{2}$$

$$\begin{aligned}
 u + \frac{d^2 u}{dr^2} &= \cancel{\frac{2u}{r}} + u + u \frac{dr^2}{r^2} + \frac{u}{r} \frac{dr^2}{r^2} \\
 &= \frac{3u}{2} \sec^2 \frac{\theta}{2} \\
 &= \frac{3u}{2} \times 2au = 3au^2
 \end{aligned}$$

$$\therefore \underline{u + \frac{d^2 u}{dr^2} = 3au^2} \quad \text{--- (1)}$$

$$\text{Now } P = \underline{h^2 u^2 \left( u + \frac{d^2 u}{dr^2} \right)}$$

$$\Rightarrow P = h^2 u^2 (3au^2) \quad (\text{From (1)})$$

$$\Rightarrow P = 3ah^2 u^4 = \frac{3ah^2}{r^4}$$

$$\Rightarrow \boxed{P \propto \frac{1}{r^4}} \text{ is the law of force reqd.}$$

At the apse,  $r = a$  and  $u = 1/a$ ;  $\frac{du}{dr} = 0$ .

$$P = \frac{3ah^2}{r^4} \quad \text{At } r = a, \quad P = \frac{3ah^2}{a^4} = \frac{3h^2}{a^3}$$

$$\text{We know that } P = 3ah^2 u^4 = h^2 u^2 \left( u + \frac{d^2 u}{dr^2} \right)$$

$$\Rightarrow \cancel{3} \cdot a u^4 = u + \frac{d^2 u}{dr^2} \quad \text{Multiplying both}$$

sides with  $2 \frac{du}{dr}$  and integrating,

~~$$\int 6au^2 \frac{du}{d\theta} d\theta = \int 2u \frac{du}{d\theta} d\theta + \frac{d}{d\theta} \left( \frac{du}{d\theta} \right)^2 d\theta$$~~

~~$$\Rightarrow 6a \times \frac{u^3}{3} = u^2 + \left( \frac{du}{d\theta} \right)^2$$~~

~~$$\Rightarrow 2au^3 = u^2 + \left( \frac{du}{d\theta} \right)^2$$~~

~~$$\Rightarrow V^2 = h^2 \left( u^2 + \left( \frac{du}{d\theta} \right)^2 \right) = 2ah^2u^3$$~~

$$V^2 = h^2 \left( u^2 + \left( \frac{du}{d\theta} \right)^2 \right). \text{ At } u = \frac{1}{a}, \frac{du}{d\theta} = 0;$$

$$V^2 = h^2 \left( \frac{1}{a^2} + 0 \right) = \frac{h^2}{a^2}$$

$$3V^2 = \frac{3h^2}{a^2}; \text{ Force at the apex is}$$

$$F = \frac{3h^2}{4a^3}$$

$$4aF = 4a \times \frac{3h^2}{4a^3} = \frac{3h^2}{a^2}$$

$$\therefore 3V^2 = 4aF = \frac{3h^2}{a^2}$$

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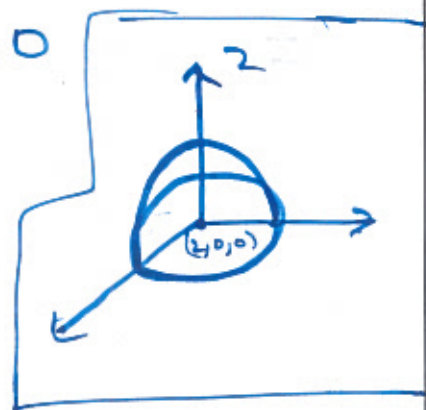


- Q.6 (c) Evaluate  $\iint_S \text{curl } A \cdot d\vec{S}$ , where  $S$  is the open surface  $x^2 + y^2 - 4x + 4z = 0, z \geq 0$  and  $A = (y^2 + z^2 - x^2)\hat{i} + (2z^2 + x^2 - y^2)\hat{j} + (x^2 + y^2 - 3z^2)\hat{k}$ . (15)

Surface S:  $x^2 + y^2 - 4x + 4z = 0$

$$\Rightarrow (x-2)^2 + y^2 = 4(1-z)$$

The curve is a paraboloid  
curving to -ve z axis.



For the interval  $z \geq 0$  it has cross section

$$(x-2)^2 + y^2 = 4 \text{ on the } xy \text{ plane (circle).}$$

By Stokes theorem, ~~there is a~~

$$\oint_C A \cdot d\vec{r} = \iint_S \text{curl } A \cdot d\vec{S}$$

Now  $\iint_S (\nabla \times A) \cdot d\vec{S} = \oint_C A \cdot d\vec{r}$  where  $C$  is

the curve ~~at z=0~~  $(x-2)^2 + y^2 = 4, z=0$ .

$$\therefore \oint_C A \cdot d\vec{r} = \oint_C ((y^2 + z^2 - x^2)\hat{i} + (2z^2 + x^2 - y^2)\hat{j} + (x^2 + y^2 - 3z^2)\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

Here  $z = dz = 0$

$$\Rightarrow \oint_C A \cdot dr = \oint (y^2 - x^2) dx + (x^2 - y^2) dy \quad \text{--- (1)}$$

$$\Rightarrow \text{Let } y = 2 \sin \theta \text{ and } x = 2 + 2 \cos \theta$$

$$\Rightarrow dy = 2 \cos \theta d\theta ; dx = -2 \sin \theta d\theta$$

$$y^2 - x^2 = (2 \sin \theta)^2 - (2 + 2 \cos \theta)^2$$

$$= 4 \sin^2 \theta - (4 + 4 \cos^2 \theta + 8 \cos \theta)$$

$$= 4 \sin^2 \theta - 4 \cos^2 \theta - 8 \cos \theta - 4$$

$$= -8 \cos^2 \theta - 8 \cos \theta$$

$$(y^2 - x^2) dx = (-8 \cos^2 \theta - 8 \cos \theta) (-2 \sin \theta) d\theta$$

$$= 16 \sin \theta (\cos^2 \theta + \cos \theta) d\theta$$

$$x^2 - y^2 = 8 \cos^2 \theta + 8 \cos \theta$$

$$(x^2 - y^2) dy = (8 \cos^2 \theta + 8 \cos \theta) (2 \cos \theta) d\theta$$

$$= 16 \cos \theta (\cos^2 \theta + \cos \theta)$$

$$\oint_C A \cdot dr = \int_0^{2\pi} 16 (\sin \theta + \cos \theta) (\cos^2 \theta + \cos \theta) d\theta$$

$$= 16 \int_0^{2\pi} \sin \theta \cos^2 \theta + \sin \theta \cos \theta + \cos^3 \theta + \cos^2 \theta d\theta$$

$$= 16 \times \int_0^{2\pi} \sin \theta \cos^2 \theta + \int_0^{2\pi} \frac{\sin 2\theta}{2} + \int_0^{2\pi} \cos^3 \theta + \int_0^{2\pi} \cos^2 \theta d\theta$$

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$$= 16 \left( 0 + 0 + 0 + 4 \times \frac{1}{2} \times \frac{\pi}{2} \right)$$

$$= \underline{\underline{16\pi}}$$

$$\therefore \iint \text{curl } A \cdot d\mathbf{s} = 16\pi$$



Q.7 (a) Solve  $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$ . (20)

Given DE:  $\frac{d^2y}{dx^2} - \left(\frac{2x+5}{x+2}\right)\frac{dy}{dx} + \left(\frac{2}{x+2}\right)y = \frac{(x+1)}{(x+2)}e^x$

Comparing with  $y'' + Py' + Qy = R$ ,

$P = -\left(\frac{2x+5}{x+2}\right)$ ;  $Q = \left(\frac{2}{x+2}\right)$ ;  $R = \frac{x+1}{x+2}e^x$

$P = -\left(\frac{2x+4}{x+2} + \frac{1}{x+2}\right) = -\left(2 + \frac{1}{x+2}\right) = -2 - \frac{1}{x+2}$

We know that if  $m^2 + Pm + Q = 0$ , then  $u = e^{mx}$  is a solution.

Taking  $m=2$ ,  $2^2 + 2P + Q$  becomes:

~~Then~~  $4 + 2P + Q = 4 + \left(-4 - \frac{2}{x+2}\right) + \left(\frac{2}{x+2}\right)$   
 $= 0$ .

$\therefore e^{2x}$  is a solution.

Let  $u = e^{2x}$ . Then other soln  $v$  is given by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u} \frac{du}{dx}\right) \frac{dv}{dx} = \frac{R}{u}.$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(-2 - \frac{1}{x+2} + \frac{2}{e^{2x}} \times 2e^{2x}\right) \frac{dv}{dx} = \left(\frac{x+1}{x+2}\right) \frac{e^x}{e^{2x}}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(2 - \frac{1}{x+2}\right) \frac{dv}{dx} = \left(\frac{x+1}{x+2}\right) e^{-x}$$

$$\text{Let } p = \frac{dv}{dx} \Rightarrow \frac{d^2v}{dx^2} = \frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx} + \left(2 - \frac{1}{x+2}\right) p = \left(\frac{x+1}{x+2}\right) e^{-x}$$

~~$\frac{dp}{dx} + \left(2 - \frac{1}{x+2}\right) p = \left(\frac{x+1}{x+2}\right) e^{-x}$~~  Integrating factor:  $IF = e^{\int k dx}$

$$\Rightarrow IF = e^{\int 2 - \frac{1}{x+2} dx} = e^{2x - \ln(x+2)} = \frac{e^{2x}}{x+2}$$

$$\Rightarrow \text{The soln is } p \times IF = \int R' \times IF + C$$

$$\Rightarrow p \times \left(\frac{e^{2x}}{x+2}\right) = \int \left(\frac{x+1}{x+2}\right) e^{-x} \times \frac{e^{2x}}{(x+2)} dx$$

$$\Rightarrow p \left(\frac{e^{2x}}{x+2}\right) = \int \frac{(x+1)e^x}{(x+2)^2} dx = \int \frac{(x+2)-1}{(x+2)^2} e^x dx$$

$$\Rightarrow p \left(\frac{e^{2x}}{x+2}\right) = \int \left(\frac{1}{x+2} - \frac{1}{(x+2)^2}\right) e^x dx$$

$$\Rightarrow p \left(\frac{e^{2x}}{x+2}\right) = \int d\left(\frac{e^x}{x+2}\right) = \frac{e^x}{x+2} + C$$

$$\Rightarrow P \left( \frac{e^{2x}}{x+2} \right) = \frac{e^x}{x+2} + C$$

$$\Rightarrow P = e^{-x} + C e^{-2x} (x+2) = \frac{dv}{dx}$$

$$\Rightarrow dv = (e^{-x} + 2C e^{-2x} + x C e^{-2x}) dx$$

Integrating on both sides,

$$v = \int dv = \int e^{-x} + 2C e^{-2x} + C x e^{-2x} dx$$

$$v = \frac{e^{-x}}{-1} + \frac{2C e^{-2x}}{-2} + C \int x e^{-2x} dx + C'$$

$$v = -e^{-x} - C e^{-2x} + C \left( x \int e^{-2x} dx - \int \frac{dx}{dx} \int e^{-2x} dx \right)$$

$$v = -e^{-x} - C e^{-2x} + C \left( \frac{x e^{-2x}}{-2} + \frac{1}{2} \int e^{-2x} dx \right) + C'$$

$$v = -e^{-x} - C e^{-2x} - \frac{C x e^{-2x}}{2} - \frac{C e^{-2x}}{4} + C'$$

$$\Rightarrow \boxed{-e^{-x} - e^{-2x} \left( \frac{5C}{4} + \frac{Cx}{2} \right) + C' = v}$$

The final solution is given by

$$\underline{\underline{y = v}}$$

$$\Rightarrow y = e^{2x} \left( -e^{-x} - e^{-2x} \left( \frac{5C}{4} + \frac{C}{2}x \right) + C' \right)$$

$$\Rightarrow \boxed{y = e^x - \frac{5C}{4} - \frac{Cx}{2} + C'e^{2x}}$$

is the reqd final solution.



Q.7 (b) Suppose  $A = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ . Evaluate  $\int_C A \cdot dr$  from  $(0,0,0)$  to  $(1,1,1)$  along the following paths C:

- $x = t, y = t^2, z = t^3$ .
- The straight lines from  $(0,0,0)$  to  $(1,0,0)$ , then to  $(1,1,0)$  and then to  $(1,1,1)$
- The straight line joining  $(0,0,0)$  and  $(1,1,1)$ . (15)

(i)  $x = t, y = t^2, z = t^3$

$\Rightarrow dx = dt, dy = 2t dt, dz = 3t^2 dt$

$$\int_C A \cdot dr = \int_C (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k} \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_0^1 (3x^2 + 6y) dx - 14yz dy + 20xz^2 dz$$

$$= \int_0^1 (3t^2 + 6t^2) - 14t^2 \times t^3 \times 2t + 20 \times t \times t^6 \times 3t^2 dt$$

$$= \int_0^1 9t^2 - 28t^6 + 60t^9 dt$$

$$= \left( \frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10} \right)_0^1 = \underline{\underline{5 \text{ units}}}$$

$$\therefore \int_C A \cdot dr = 5 \text{ units}$$

(ii) (A) path  $(0,0,0)$  to  $(1,0,0)$

$\Rightarrow y = dy = 0$  and  $z = dz = 0$ .

$$\Rightarrow \int_C A \cdot dr = \int_0^1 (3x^2 + 6y) dx = \int_0^1 3x^2 dx$$

$$= 1 \text{ unit}$$

② path  $(0,0)$  to  $(1,0)$

Here,  $x=1$ ,  $dx=0$ ;  $dz=z=0$

$$\Rightarrow \int_{C_1} A \cdot dr = \int_{C_1} -14yz dy = \underline{0} \quad (\because z=0)$$

③ path  $(1,0)$  to  $(1,1)$

Here,  $x=y=1$ ;  $dx=dy=0$ .

$$\Rightarrow \int_{C_2} A \cdot dr = \int_{C_2} 20xz^2 dz = \int_0^1 20z^2 dz = \underline{\underline{\frac{20}{3} \text{ units}}}$$

$$\Rightarrow \int_C A \cdot dr = \int_{C_1} + \int_{C_2} + \int_{C_3} = 1 + 0 + \frac{20}{3} = \underline{\underline{\frac{23}{3} \text{ units}}}$$

(iii) straight line joining  $(0,0,0)$  to  $(1,1,1)$

is given by  $x=y=z \Rightarrow dx=dy=dz$

$$\Rightarrow \int A \cdot dr = \int (3x^2+6y) dx - 14dy yz + 20xz^2 dz$$

Let  $x=y=z=k$  and  $dx=dy=dz=dk$ .

$$\Rightarrow \int A \cdot dr = \int 3k^2 + 6k - 14k^2 + 20k^3 dk$$

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$$\Rightarrow \int_C A \cdot dr = \int_0^1 (20k^3 - 11k^2 + 6k)$$

$$= \left( \frac{20k^4}{4} - \frac{11k^3}{3} + \frac{6k^2}{2} \right)_0^1$$

$$= \underline{\underline{\frac{13}{3} \text{ units}}}$$

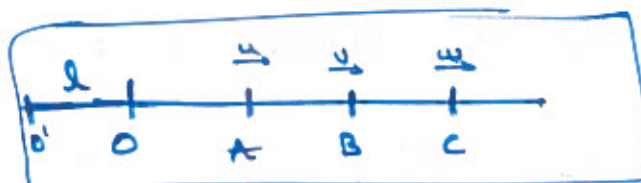
$$\therefore \int_C A \cdot dr = \frac{13}{3} \text{ units}$$

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Q.7 (c)

If in a simple harmonic motion  $u, v, w$  be the velocities at a distance  $a, b, c$  from a fixed point on the straight line (which is not the centre of force), show that the period  $T$  is given

by the equation  $\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$  (15)



Let the fixed point be  $O$ . Then  $OA = a$ ,  
 $OB = b$ ,  $OC = c$ .

The velocity in case of SHM is given by

$v^2 = \omega^2(A^2 - x^2)$ . Let  $O'$  be the mean position and  $OO' = l$

~~At  $x = a$ ,  $u^2 = \omega^2(A^2 - a^2)$~~   
~~At  $x = b$ ,  $v^2 = \omega^2(A^2 - b^2)$~~   
~~At  $x = c$ ,  $w^2 = \omega^2(A^2 - c^2)$~~

At the point A;

$$u^2 = \omega^2(A^2 - (l+a)^2) \quad \text{--- (1)}$$

$$\Rightarrow \frac{u^2}{\omega^2} = A^2 - l^2 - a^2 - 2al$$

$$\Rightarrow A^2 + 2al + \frac{u^2}{\omega^2} = 0 \quad \text{--- (2)}$$

At the point B;

$$v^2 = \omega^2(A^2 - (l+b)^2)$$

$$\Rightarrow A^2 + 2bl + \frac{v^2}{\omega^2} = 0 \quad \text{--- (3)}$$

Similarly at the point C,

$$l^2 - A^2 + 2clc^2 + \frac{w^2}{w^2} = 0 \text{ --- (4)}$$

Writing the equations in matrix form,

$$\begin{bmatrix} 1 & a & \frac{u^2}{w^2} - a^2 \\ 1 & b & \frac{v^2}{w^2} - b^2 \\ 1 & c & \frac{w^2}{w^2} - c^2 \end{bmatrix} \begin{bmatrix} l^2 - A^2 \\ 2al \\ 1 \end{bmatrix} = 0$$

Taking determinant,

$$\begin{vmatrix} 1 & a & \frac{u^2}{w^2} - a^2 \\ 1 & b & \frac{v^2}{w^2} - b^2 \\ 1 & c & \frac{w^2}{w^2} - c^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & a & \frac{u^2}{w^2} \\ 1 & b & \frac{v^2}{w^2} \\ 1 & c & \frac{w^2}{w^2} \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \frac{u^2}{w^2} & \frac{v^2}{w^2} & \frac{w^2}{w^2} \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow \frac{1}{w^2} \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$\Rightarrow \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \omega^2 (a-b)(b-c)(c-a)$$

We know that  ~~$\omega = \frac{2\pi}{T}$~~   $T = \frac{2\pi}{\omega}$

$$\Rightarrow \omega = \frac{2\pi}{T} \Rightarrow \omega^2 = \frac{4\pi^2}{T^2}$$

$$\Rightarrow \frac{4\pi^2}{T^2} (a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

Hence proved.



Q.8 (a) Solve by the method of variation of parameters the differential equation

$$x''(t) - 2\frac{x(t)}{t^2} = t, \text{ where } 0 < t < \infty. \quad (20)$$

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Q.8 (b)

Suppose a cylinder of any cross section is balanced on another fixed cylinder, the contact of curved surface being rough and the common tangent line horizontal. Let  $\rho$  and  $\rho'$  be radii of curvature of the two cylinders at the point of contact and  $h$  be the height of centre of gravity of the upper cylinder above the point of contact. Show that the upper cylinder is

balanced in stable equilibrium if  $h < \frac{\rho\rho'}{\rho + \rho'}$ . (15)

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Q.8 (c)

A uniform ladder of 10m length and of 10kg weight rests with its foot on the rough ground and its upper end against a smooth wall, the inclination to the vertical being  $\alpha$ . A force  $P$  is applied horizontally to the ladder at a point distance 3m from the foot, so as to make the

foot approach the wall. Prove that the force  $P$  must exceed  $\frac{100}{7} \left( \mu + \frac{1}{2} \tan \alpha \right)$ , where  $\mu$  is the coefficient of friction at the foot. (15)



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SPACE FOR ROUGH WORK

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$\frac{u^2}{w^2} \rightarrow a^2 = *$

$$\begin{vmatrix} 1 & a & \frac{u^2}{w^2} - a^2 \\ 1 & b & \frac{u^2}{w^2} - b^2 \\ 1 & c & \frac{u^2}{w^2} - c^2 \end{vmatrix} = 0$$

$$u = \frac{1}{a} + \frac{1}{(1-\cos\theta)}$$

$$\frac{du}{d\theta} = \frac{1}{a} + \frac{-1}{(1-\cos\theta)^2} \times \sin\theta \cdot \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \frac{-\sin\theta}{a(1-\cos\theta)^2} = -\frac{u \sin\theta}{1-\cos\theta}$$

$$bc(c-b) - a(c-b)(c+b) + a^2(c-b) = (bc^2 - b^2c) - a(c^2 - b^2) + a^2(c-b)$$

$$u = \frac{1}{2a} \operatorname{cosec}^2\left(\frac{\theta}{2}\right)$$

$$= (c-b) \left( bc - ac - ab + a^2 \right)$$

$$\frac{du}{d\theta} = \frac{1}{2a} \times \left( -\operatorname{cosec}^2\left(\frac{\theta}{2}\right) \cot\left(\frac{\theta}{2}\right) \right)$$

$$= (c-b) \left( \frac{b(c-a) - a(c-a)}{(b-a)(c-a)(c-b)} \right) = -\frac{1}{2a} \operatorname{cosec}^2\left(\frac{\theta}{2}\right) \cot\left(\frac{\theta}{2}\right)$$

$$\frac{u}{\delta/2} = \frac{1}{\delta} = -u \cot\left(\frac{\theta}{2}\right)$$

$$(a-b)(b-c)(c-a)$$

$$\frac{q}{2a} = \begin{vmatrix} 1 & 2a & \frac{u^2}{w^2} - a^2 \\ 1 & 2b & \frac{u^2}{w^2} - b^2 \\ 1 & 2c & \frac{u^2}{w^2} - c^2 \end{vmatrix} = 0$$

$$\frac{6916}{25 \times 18} + \frac{877}{25 \times 18}$$

$dx = dr \cos\theta$

$0 = \frac{1}{\delta} \begin{bmatrix} 2a & u \\ 2b & u \\ 2c & u \end{bmatrix}$

$x = (2 + 2\cos\theta)$

$(x-2)^2 + y^2 = 4$

$d(x^2 + y^2)$