

NEXT IAS

(To be filled by candidate)

Name of Candidate : MADHUKANT

Roll No. : Date of Examination : 06/08/24

Mobile Number E-mail

Exam Centre : Old Rajinder Nagar ☐

Online ☒

CSE (MAINS) TEST SERIES-2024

Test-04

MATHEMATICS
OPTIONAL

Dated : 14-07-2024

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are **EIGHT** questions divided in **Two sections** and printed in **ENGLISH**.

Candidate has to attempt **FIVE** questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

DELHI CENTRE:

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	20			
2. (b)	18	15			
2. (c)	21	15			
3. (a)	24	20			
3. (b)	27	15			
3. (c)	30	15			
4. (a)	34	15			
4. (b)	37	20			
4. (c)	40	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	15			
6. (b)	57	15			
6. (c)	60	20			
7. (a)	64	15			
7. (b)	67	15			
7. (c)	70	20			
8. (a)	74	20			
8. (b)	78	15			
8. (c)	81	15			
Grand Total					

Remarks

Observations:

SECTION-A

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- Q.1 (a) Formulate PDE for surfaces whose tangent planes form a tetrahedron of constant volume with coordinate planes. (10)

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Q.1 (b) State the sufficient condition for convergence of the Gauss Seidel iteration method. (10)

Let $x_1, x_2, \dots, x_n$ be n variable satisfying following linear eqⁿ $\rightarrow$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = d_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = d_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = d_n$$

without loss of generality, let us assume,

above equation are in partial pivoting. if not we can rearrange them.

Now for Gauss Seidel iteration method to converge,

$$|a_{11}| > |a_{12}| + |a_{13}| + \dots + |a_{1n}|$$

$$|a_{22}| > |a_{21}| + |a_{23}| + \dots + |a_{2n}|$$

$$\vdots$$

$$|a_{nn}| > |a_{n1}| + |a_{n2}| + \dots + |a_{n(n-1)}|$$

— is sufficient condition for convergence

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Q.1 (c)

Express the Boolean function $f(x, y, z) = x + \overline{xy} + \overline{xz} + z$ in disjunctive normal form (DNF) and construct the truth table for the function. (10)

Given boolean function:

$$f(x, y, z) = x + \overline{xy} + \overline{xz} + z$$

$$= x + \overline{xy} \cdot \overline{xz} + z \quad (\text{De Morgan's law})$$

$$= x + xy \cdot (\overline{x} + \overline{z}) + z \quad (\text{De Morgan's in } \overline{xz} = \overline{x} + \overline{z})$$

$$= x + xyx + xy\overline{z} + z \quad (\text{distributive})$$

$$= x + xy + xy\overline{z} + z \quad (\text{Associative and Idempotent})$$

$$= x(1+y) + z + \overline{z}xy \quad (\text{distributive})$$

$x(1+y) = x + xy$

$$= x + z + \overline{z}(xy) \quad (\text{dominance})$$

$1+y = 1$

$$= x + z + xy \quad (\because a + \overline{a}b = a + b)$$

$$= x + xy + z$$

$$= x(1+y) + z$$

$$= x + z$$

(distributive and dominance)

$\therefore f(x, y, z) = x + z$ is in disjunctive normal form

Candidates
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on this marginTruth table for $f(x, y, z)$

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Q.1 (d)

A rod of length $2a$ revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α . Prove that the direction of reaction at the hinge makes with the vertical, an angle

$$\tan^{-1} \left[\frac{3}{4} \tan \alpha \right].$$

(10)

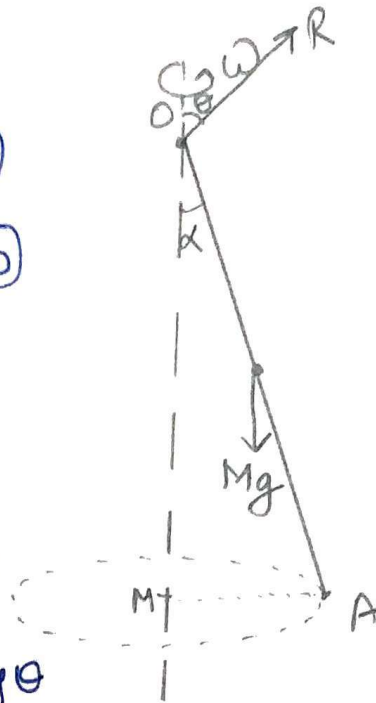
Let OA be the rod of length $(2a)$

rotating with angular velocity (ω) about OM .

Given: $\angle MOA = \alpha$

To find: Reaction & its direction

Let Reaction at hinge be R making θ with vertical.



Balancing Vertical forces.

$$R \cos \theta = Mg \quad \text{--- (I)}$$

Balancing horizontal forces $\rightarrow$

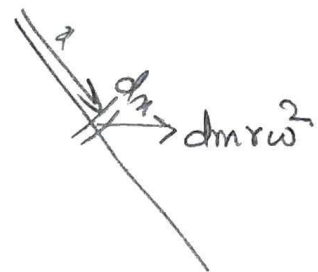
Centripetal force due to rotation

$$= \int_{x=0}^{2a} dm \, r \omega^2$$

$$= \int_{x=0}^{2a} \frac{M}{2a} dx (x \sin \alpha) \omega^2$$

$$= \int_{x=0}^{2a} \frac{M(\sin \alpha) \omega^2}{2a} x dx = M \omega^2 a \sin \alpha$$

$$\therefore R \sin \theta = M \omega^2 a \sin \alpha \quad \text{--- (II)}$$



Now, $\tan \theta = \frac{\omega^2 a \sin \alpha}{g}$ — (III)

Now, Moment of forces about O $\rightarrow$

Moment of gravitational abt O = $Mg(a \sin \alpha)$.

Moment of centripetal = $\int (dm r \omega^2) x \cos \alpha$

$= \int_{x=0}^{2a} \frac{M}{2a} dx (x \sin \alpha) \omega^2 x \cos \alpha$

$= \frac{M \omega^2 \sin \alpha \cos \alpha}{2a} \int_{x=0}^{2a} x^2 dx = \frac{M \omega^2 \sin \alpha \cos \alpha}{2a} \left[\frac{x^3}{3} \right]_0^{2a}$

$= \frac{M \omega^2 \sin \alpha \cos \alpha}{2a} \left(\frac{8a^3}{3} \right) = \frac{4}{3} M \omega^2 a^2 \sin \alpha \cos \alpha$

Moment balance $\rightarrow$

$Mg(a \sin \alpha) = \frac{4}{3} M \omega^2 a^2 \sin \alpha \cos \alpha$

$\frac{\omega^2 a}{g} = \frac{3}{4} \cos \alpha$

$\therefore$ from eq (III),

$\tan \theta = \frac{\omega^2 a \sin \alpha}{g} = \frac{3}{4} \sin \alpha = \frac{3}{4} \tan \alpha$

$\Rightarrow \theta = \tan^{-1} \left(\frac{3}{4} \tan \alpha \right)$

Q.1 (e) Find the equation of the streamlines for the flow $\mathbf{q} = -(3y^2)\hat{i} - 6x\hat{j}$ at the point (1, 1). (10)

Given velocity: $\vec{q} = -3y^2\hat{i} - 6x\hat{j}$

velocity component $u = -3y^2$ $v = -6x$

then, equation of streamlines is given by

$$\frac{dx}{u} = \frac{dy}{v}$$

$$\Rightarrow \frac{dx}{-3y^2} = \frac{dy}{-6x} \Rightarrow \frac{dx}{y^2} = \frac{dy}{2x}$$

$$\Rightarrow 2x dx - y^2 dy = 0$$

$$\Rightarrow x^2 - \frac{y^3}{3} = C$$

at (1, 1) $C = 2/3$

$\Rightarrow$ equation of streamline at (1, 1) for given $\vec{q}$

is $\boxed{x^2 - \frac{y^3}{3} = \frac{2}{3}}$

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Q.2 (a) Find the general solution and singular solution of $6yz - 6pxy - 3qy^2 + pq = 0$. (20)

Given p.d.e is: $6yz - 6pxy - 3qy^2 + pq = 0$

$$\text{or } z - px - \frac{3qy^2}{6y} + \frac{pq}{6y} = 0$$

$$\Rightarrow z = px + \frac{qy}{2} - \frac{pq}{6y}$$

let $y^2 = Y$

then $2y \partial y = \partial Y$

$$\text{then, } z = \frac{\partial z}{\partial x} \cdot x + \frac{\partial z}{\partial y (2y)} y^2 - \frac{pq}{3 \cdot 2y}$$

let $\frac{\partial z}{\partial Y} = \frac{\partial z}{2y \partial y} = Q$

$$\text{then, } z = px + QY - \frac{pQ}{3}$$

this is in clairaut's form

$\therefore$ general solution to this will be

when $p = a$ & $Q = b$, a, b are
arbitrary const.

∴ general solⁿ →

$$z = ax + by - \frac{ab}{3}$$

or

$$\boxed{z = ax + by^2 - \frac{ab}{3}} \text{ g.s. — ①}$$

for Singular solution →

we differentiate eq ① with a & b and

eliminate them

diff w.r.t. a →

$$x - \frac{b}{3} = 0 \quad \text{and}$$

diff w.r.t. b →

$$y^2 - \frac{a}{3} = 0$$

$$\Rightarrow a = 3y^2$$

$$\Rightarrow b = 3x$$

On substituting value of a, b in eq ①

$$z = (3y^2)x + (3x)y^2 - \frac{(3y^2)(3x)}{3}$$

$$= 3xy^2 + 3xy^2 - 3xy^2$$

$$\Rightarrow \boxed{z = 3xy^2} \text{ is } \underline{\text{Singular Solution}}$$

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Q.2 (b)

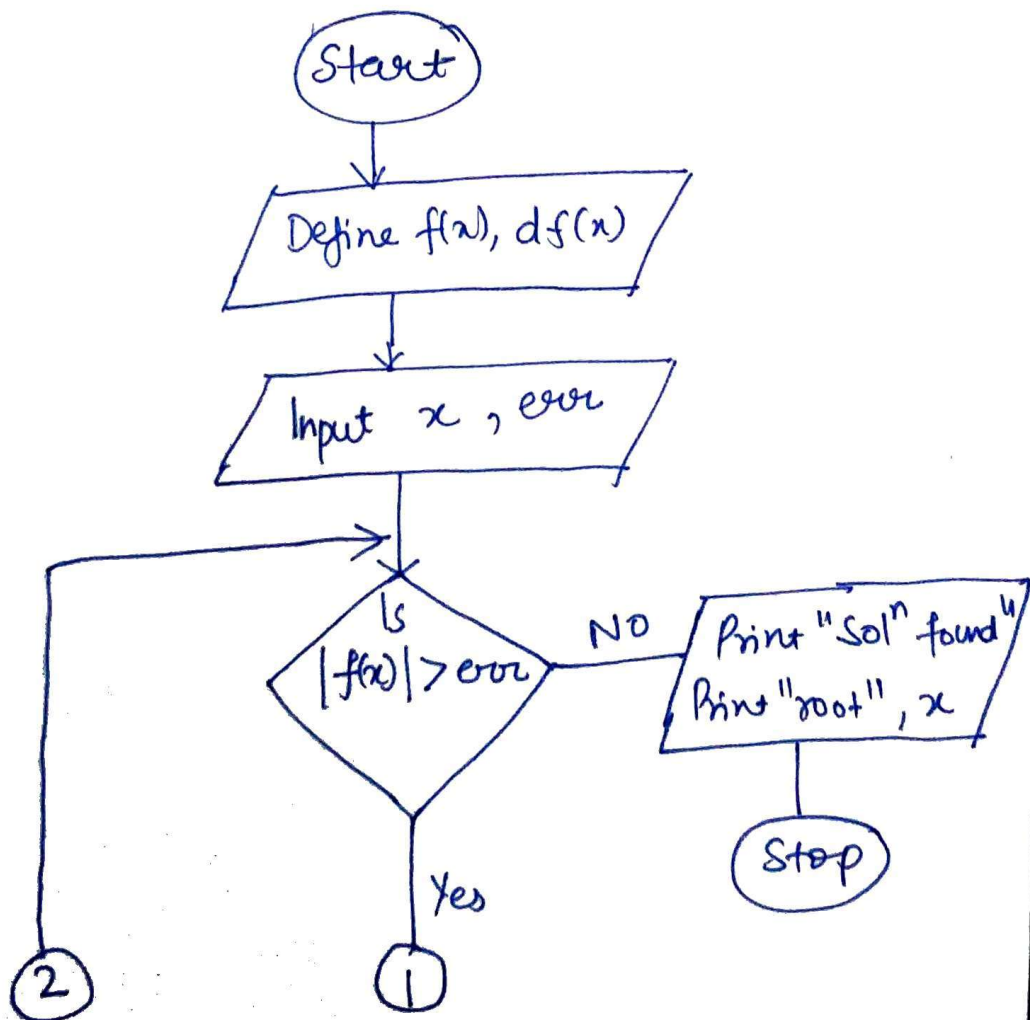
Write an algorithm in the form of a flow chart for Newton-Raphson method. Describe the cases of failure of this method. (15)

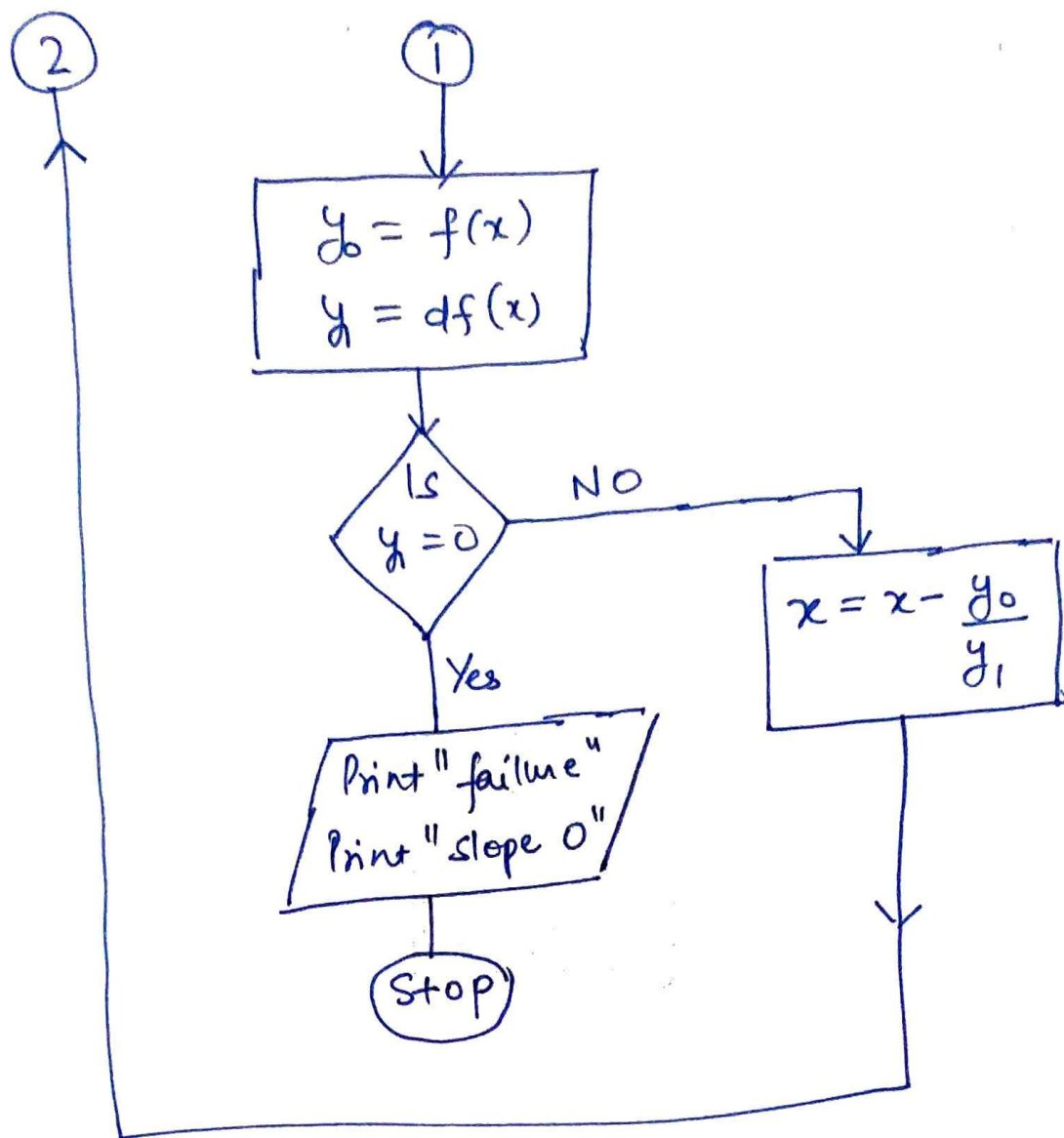
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In Newton Raphson,

$$x_0 = x - \frac{f(x)}{f'(x)}$$

let us assume, we iterate upto condition that $f(x_0) < \text{error}$, where error is ~~for~~ max^m error.





Cases of failure :

when slope become 0 - i.e. $f'(x) = 0$

for some intermediary point.

Q.2 (c) Show that the surface $\frac{x^2}{a^2 k^2 t^4} + k t^2 \left(\frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = 1$ is a possible form of boundary surface of a liquid at time t . (15)

let given surface be $\rightarrow$

$$F(x, y, z, t) = \frac{x^2}{a^2 k^2 t^4} + k t^2 \left(\frac{y^2}{b^2} + \frac{z^2}{c^2} \right) - 1 = 0$$

Boundary condition for fluid is $\rightarrow$

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} = 0$$

Now,

$$\begin{aligned} \frac{DF}{Dt} &= \frac{-4x^2}{a^2 k^2 t^5} + 2kt \left(\frac{y^2}{b^2} + \frac{z^2}{c^2} \right) \\ &\quad + u \left(\frac{2x}{a^2 k^2 t^4} \right) + v \left(kt^2 \cdot \frac{2y}{b^2} \right) + w \left(kt^2 \cdot \frac{2z}{c^2} \right) \end{aligned}$$

$$\begin{aligned} &= \frac{2x}{a^2 k^2 t^4} \left(u - \frac{2x}{t} \right) + \frac{kt y}{b^2} (2tv + 2y) \\ &\quad + \frac{2ktz}{c^2} (tw + z) \end{aligned}$$

$\therefore x, y, z$ are independent, taking x, y, z component to 0.

$$u = \frac{2x}{t}$$

$$v = -\frac{y}{t}$$

$$w = -\frac{z}{t}$$

$$\frac{dx}{dt} = \frac{2x}{t}$$

$$\frac{dy}{dt} = -\frac{y}{t}$$

$$\frac{dz}{dt} = -\frac{z}{t}$$

On integrating $\rightarrow$

$$x = c_1 t^2$$

$$y = \frac{c_2}{t}$$

$$z = \frac{c_3}{t}$$

Now, checking if u, v, w corresponds to a fluid motion by equation of Continuity $\rightarrow$

$$\text{i.e. } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$= \frac{2}{t} - \frac{1}{t} - \frac{1}{t} = 0$$

$\therefore$ equation of Continuity is satisfied

$\Rightarrow$ given F is possible form of boundary

surface of liquid.

SECTION-B

- Q.5 (a) Transform the equation $yz_x - xz_y = 0$ into one in polar coordinates and thereby show that the solution of the given equation represents surfaces of revolution. (10)

Given equation: $yz_x - xz_y = 0$

In polar form, $x = r \cos \theta$ $y = r \sin \theta$
 $r^2 = x^2 + y^2$

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$z_x = \cos \theta \frac{\partial z}{\partial r}$$

$$z_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r} \sin \theta + \frac{\partial z}{\partial \theta} \cdot \frac{1}{r}$$

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- Q.5 (b) Verify that the equation $yz(y+z)dx + xz(x+z)dy + xy(x+y)dz = 0$ is integrable and find its solution. (10)

Given equation: $yz(y+z)dx + xz(x+z)dy + xy(x+y)dz = 0$

or,

$$(y^2zdx + yz^2dx) + (x^2zdy + xz^2dy)$$

$$+ (x^2ydz + xy^2dz) = 0$$

$$z^2(ydx + xdy) + y^2(zdx + xdz)$$

$$+ x^2(zdy + ydz) = 0$$

$$\Rightarrow z^2 d(xy) + y^2 d(xz) + x^2 d(yz) = 0$$

dividing whole eqⁿ by $x^2y^2z^2$

$$\frac{d(xy)}{(xy)^2} + \frac{d(xz)}{(xz)^2} + \frac{d(yz)}{(yz)^2} = 0$$

$\therefore$ the given equation is integrable

and solution will be $\rightarrow$

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$$\int \frac{d(xy)}{(xy)^2} + \frac{d(yz)}{(yz)^2} + \frac{d(xz)}{(xz)^2} = \int 0$$

$$\Rightarrow \boxed{\frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz} = C}$$

- Q.5 (c) Express the Boolean function $F(x, y, z) = xy + x'z$ in a product of maxterm form. (10)

Given boolean function:

$$F(x, y, z) = xy + \bar{x}z$$

writing truth table for $F(x, y, z)$

x	y	z	xy	$\bar{x}z$	$xy + \bar{x}z$	
0	0	0	0	0	0	M_0
0	0	1	0	1	1	
0	1	0	0	0	0	M_2
0	1	1	0	1	1	
1	0	0	0	0	0	M_3
1	0	1	0	0	0	M_4
1	1	0	1	0	1	
1	1	1	1	0	1	

∴ Product of Maxterm form will be

$$= \Pi(0, 2, 3, 4)$$

$$= (x+y+z) \cdot (x+\bar{y}+z) \cdot (\bar{x}+y+z) \cdot (\bar{x}+y+\bar{z})$$

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Q.5 (d)

Convert $(3798.3875)_{10}$ into octal and hexadecimal equivalents.

(10)

To Convert : $(3798.3875)_{10}$

① Octal

$\div$		Remainder
8	3798	6
8	474	2
8	59	3
8	7	7
	0	

$\times$		
8	.3875	~~2~~
8	<u>3</u> .1	3
8	<u>0</u> .8	0
8	<u>6</u> .4	6
8	<u>3</u> .2	3
8	<u>1</u> .6	1
8	4.8	4

$\therefore (3798.3875)_{10} =$

$(7326.306314)_8$

① Hexadecimal $(3798.3875)_{10}$

		Remainder
16	3798	6
16	237	13 = D
16	14	14 = E
	0	

↑

16	.3875	
16	<u>6</u> .2	6
16	<u>3</u> .2	3
16	<u>3</u> .2	3

↓

∴ In Hexadecimal form →

$$(3798.3875)_{10} = (ED6.633\cdots)_{16}$$

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Q.5 (e) Find the moment of inertia of triangular lamina.

(10)

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- Q.7 (a) Find the family of surfaces orthogonal to the family of surfaces given by the differential equation $(y+z)p + (z+x)q = x+y$. (15)

Given differential equation :

$$(y+z)p + (z+x)q = x+y \quad \text{--- (1)}$$

We know that family of surfaces orthogonal

to $Pp + Qq = R$ is given by

--- (2)

differential eqⁿ →

$$Pdx + Qdy + Rdz = 0$$

On comparing eqⁿ (1) & (2)

$$P = y+z \quad Q = z+x \quad R = x+y$$

∴ p.d.e. of orthogonal surface —

$$(y+z)dx + (z+x)dy + (x+y)dz = 0$$

$$(ydx + xdy) + (zdx + xdz) + (zdy + ydz) = 0$$

$$\rightarrow d(xy) + d(xz) + d(yz) = 0$$

On Integrating $\boxed{xy + yz + zx = C}$

required family of orthogonal Surfaces.

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2.7 (b)

Using Runge Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$.

Use four decimal places for calculation and step length 0.2. (15)

Given: $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, $y(0) = 1$
 $= f(x, y)$ (say)

to find $y(0.2) = ?$ using Runge Kutta Method
 of 4th order.

let $x_0 = 0$, $y_0 = y(0) = 1$

& $h = 0.2$ (step length)

then for Runge Kutta Method

~~$K_1 = h(f(x_0 + h, y_0 + K_1))$~~

$K_1 = h f(x_0, y_0)$

$K_2 = \frac{h}{2} f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$K_3 = \frac{h}{2} f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$

$K_4 = h f(x_0 + h, y_0 + K_3)$

& $K = \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$

& $y(0.2) = y(x_0 + h) = y_0 + K$

$$\begin{aligned}\text{Now, } K_1 &= h f(x_0, y_0) = (0.2) f(0, 1) \\ &= (0.2) f_1 = 0.2\end{aligned}$$

$$K_2 = (0.2) f(0.1, 1+0.1) = (0.2) \left(\frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2} \right)$$

$$= 0.0984$$

$$K_3 = 0.1 f(0.1, 1+0.0492)$$

$$= 0.0982$$

$$K_4 = 0.2 f(0.1, 1)$$

$$K_2 = 0.1967$$

$$K_3 = (0.2) f(0.1, 1.09835)$$

$$= 0.19671$$

$$K_4 = (0.2) f(0.1, 1.09835)$$

$$= 0.19671$$

$$K = \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 0.19732$$

$$\therefore \boxed{y(0.2) = 1.1973}$$

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Q.7 (c) For a dynamical system having two degrees of freedom, the Lagrangian is given by

$$L = \frac{m}{2} \left(a^2 \dot{q}_1^2 + \dot{q}_2^2 \right) - \frac{k}{2} (a^2 + q_2^2), \text{ where } q_1 \text{ and } q_2 \text{ are generalized coordinates. Find the}$$

corresponding Hamiltonian and derive the Hamiltonian equations of motion. Show further that the generalized momentum corresponding to q_1 is constant. Show that the system exhibits a simple harmonic motion with respect to the generalized coordinate q_2 . (20)

Lagrangian of system is given by -

$$L = \frac{m}{2} (a^2 \dot{q}_1^2 + \dot{q}_2^2) - \frac{k}{2} (a^2 + q_2^2)$$

$$= T - V$$

Corresponding Hamiltonian will be

$$H = T + V$$

$$= \frac{m}{2} (a^2 \dot{q}_1^2 + \dot{q}_2^2) + \frac{k}{2} (a^2 + q_2^2)$$

$$\text{Let } p_{q_1} = \frac{\partial L}{\partial \dot{q}_1} = ma^2 \dot{q}_1$$

$$\& \quad p_{q_2} = \frac{\partial L}{\partial \dot{q}_2} = m \dot{q}_2$$

$$\Rightarrow \dot{q}_1 = \frac{p_{q_1}}{ma^2} \quad \dot{q}_2 = \frac{p_{q_2}}{m}$$

Replacing in Hamiltonian eqⁿ.

$$H = \frac{m}{2} \left(\frac{p_{q_1}^2}{m^2 a^2} + \frac{p_{q_2}^2}{m^2} \right) + \frac{k}{2} (a^2 + q_2^2)$$

$$H = \frac{p_{q_1}^2}{2ma^2} + \frac{p_{q_2}^2}{2m} + \frac{k}{2} (a^2 + q_2^2)$$

Hamiltonian eqⁿ for $q_1 \rightarrow$

$$\dot{q}_1 = \frac{\partial H}{\partial p_{q_1}} \quad \dot{p}_{q_1} = -\frac{\partial H}{\partial q_1}$$

$$\dot{q}_1 = \frac{p_{q_1}}{ma^2} \quad \dot{p}_{q_1} = 0$$

$$\Rightarrow \ddot{q}_1 = \frac{\dot{p}_{q_1}}{ma^2} = 0$$

$$\Rightarrow \dot{q}_1 = \text{const}$$

$\rightarrow$ generalised momentum i.e. $\boxed{m\dot{q}_1 = \text{const.}}$
is constant

Hamiltonian eqⁿ for $q_2 \rightarrow$

$$\dot{q}_2 = \frac{\partial H}{\partial p_{q_2}} \quad \dot{p}_{q_2} = -\frac{\partial H}{\partial q_2}$$

$$\dot{q}_2 = \frac{p_{q_2}}{m} \quad \dot{p}_{q_2} = -kq_2$$

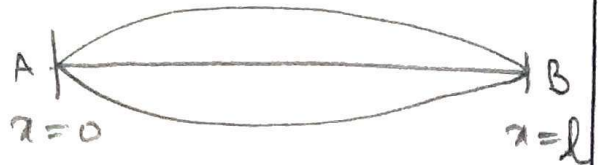
$$\ddot{q}_2 = -\frac{kq_2}{m}$$

$$\Rightarrow \boxed{\ddot{q}_2 = -\left(\frac{k}{m}\right) q_2}$$

is eqⁿ of SHM in q_2

- Q.8 (a) A tightly stretched string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $f(x) = \mu x(l-x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at time $t > 0$. (20)

let String AB be stretched $AB = l = 1$
& fixed at $x=0$ (A) & $x=l$ (B).



At time $t=0$,

shape is given by $f(x) = \mu x(l-x)$

let $y(x, t)$ be displacement of any point x
of string at time t .

Now, we know that eqⁿ $y(x, t)$ will satisfy
following partial differential eqⁿ.

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}} \quad \text{where } c \text{ is a constant} \\ \text{--- (1) for string.}$$

Now, we use Separation of Variable $\rightarrow$

$$\text{let } y(x, t) = X(x) \cdot T(t) \quad \text{--- (1)}$$

then eq (1) can be written as

$$X'' T = \frac{1}{c^2} T'' X$$

$$\text{or } \frac{X''}{X} = \frac{T''}{c^2 T} = a(\text{some const.}).$$

Case I: if $a=0$

$$X''=0 \Rightarrow X(x)=ax+b$$

$$y(x,t) = T(ax+b) \quad u(0,t)=0 \Rightarrow b=0$$

$$u(l,t)=0 \Rightarrow a=0$$

$\Rightarrow y(x,t)=0$ which is Non-feasible solution

Case II: if $a>0$ let $a=m^2$

$$X'' - m^2 X = 0$$

$$\Rightarrow X = C_1 e^{mx} + C_2 e^{-mx}$$

$$y(x,t) = T(C_1 e^{mx} + C_2 e^{-mx})$$

$$y(0,t)=0 \Rightarrow C_1 + C_2 = 0 \Rightarrow C_2 = -C_1$$

$$y(l,t)=0 \Rightarrow C_1 e^{ml} + C_2 e^{-ml} = 0$$

$$C_1 (e^{ml} - e^{-ml}) = 0$$

$$\Rightarrow C_1 = 0 \Rightarrow C_2 = 0$$

$\Rightarrow y(l,t)=0$ which is not feasible solution

Case III: if $a<0$

$$\text{let } a = -m^2$$

Now, $X'' + m^2 X = 0$ & $T'' + c^2 m^2 T = 0$

Consider $X'' + m^2 X = 0$
 $(D^2 + m^2)X = 0$

Solⁿ for $X = C_1 \cos mx + C_2 \sin mx$

lly, for $T \rightarrow T = C_3 \cos mct + C_4 \sin mct$

Now, $y(x, t) = (C_1 \cos mx + C_2 \sin mx) \times$
 $(C_3 \cos mct + C_4 \sin mct)$

$y(0, t) = 0 \Rightarrow C_1 = 0$

$y(l, t) = 0 \Rightarrow \sin ml = 0 \Rightarrow ml = n\pi$
 $\Rightarrow m = \frac{n\pi}{l}$

$\therefore y(x, t) = C_2 \sin \frac{n\pi x}{l} \left(C_3 \cos \frac{n\pi c}{l} t + C_4 \sin \frac{n\pi c}{l} t \right)$

$\therefore$ there will be many values of n

$\therefore$ a general solution will be

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \left[A_n \cos\left(\frac{n\pi c}{l} t\right) + B_n \sin\left(\frac{n\pi c}{l} t\right) \right]$$

$\therefore$ at $t=0$, string is released.

$\Rightarrow dy'(x, 0) = 0$

$$\Rightarrow \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) [A_n(0) + B_n(1)] = 0 \Rightarrow B_n = 0$$

$$\Rightarrow \boxed{y(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi c}{l} t\right)}$$

$$\text{At } t=0, y(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$$

which is fourier series

$$\therefore A_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2\mu}{l} \int_0^l x(l-x) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{2\mu}{l} \left[0 + \int_0^l \frac{(l-2x) \cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} dx \right]$$

$$= \frac{2\mu}{l} \left[\frac{(l-2x) \sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} \right]_0^l - \int_0^l \frac{(-2) \sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} dx$$

$$= \frac{2\mu}{l} \left[\frac{2 \cdot (-\cos\left(\frac{n\pi x}{l}\right))}{\left(\frac{n\pi}{l}\right)^3} \right]_0^l = \boxed{\frac{4\mu l^2 (1 - \cos n\pi)}{n^3 \pi^3}}$$

$$\text{when } n = \text{odd } A_n = \frac{8\mu l^2}{n^3 \pi^3} \quad n = \text{even}, A_n = 0.$$

$$\therefore \boxed{y(x, t) = \sum_{m=0}^{\infty} A_{2m+1} \sin\left(\frac{(2m+1)\pi x}{l}\right) \cos\left(\frac{(2m+1)\pi c t}{l}\right)}$$

$$\text{where, } A_{2m+1} = \frac{8\mu l^2}{(2m+1)^3 \pi^3}$$

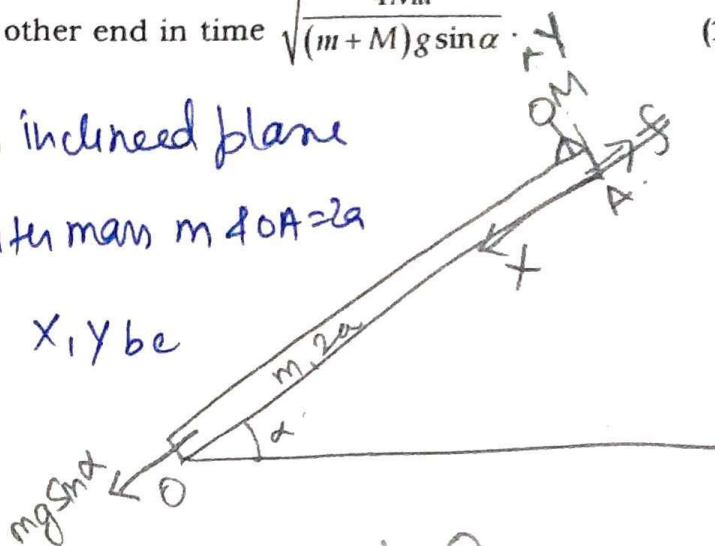
- Q.8 (b) A plank of mass m and length $2a$ is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, a man of mass M , starting from the upper end walks down the plank so that it does not move.

show that he will reach the other end in time $\sqrt{\frac{4Ma}{(m+M)g\sin\alpha}}$. (15)

let OA be plank on inclined plane
of inclination α with mass m & OA = $2a$

Man of mass M . let x, y be

Coordinate axes as
shown in figure



then Horizontal force $\rightarrow$

then balancing
force along x-axis for plank

$$m\ddot{x}_1 = mg\sin\alpha - f$$

but, $\therefore$ plank is not moving $\ddot{x}_1 = 0$

$$\Rightarrow \underline{f = mg\sin\alpha} \quad \text{--- (1)}$$

Now balancing force along x-axis for Man

$$M\ddot{x} = Mg\sin\alpha + f$$

$$= Mg\sin\alpha + mg\sin\alpha = (M+m)g\sin\alpha$$

$$\ddot{x} = \frac{(M+m)g\sin\alpha}{M}$$

∴ acceleration of Man in x -direction is

$$\ddot{x} = \frac{(M+m)g \sin \alpha}{M}$$

∴ Man is starting from rest i.e. initial velocity = 0

∴ using $s = ut + \frac{1}{2} \ddot{x} t^2$

$$2a = 0 + \frac{1}{2} \left(\frac{(M+m)g \sin \alpha}{M} \right) t^2$$

$$\Rightarrow t = \sqrt{\frac{4Ma}{(M+m)g \sin \alpha}}$$

Ans

- Q.8 (c) A two-dimensional flow field is given by $\psi = xy$.
- Show that the flow is irrotational.
 - Find the velocity potential.
 - Verify that ψ and ϕ satisfy the Laplace equation
 - Find the streamlines and potential lines. (15)

Given: flow field given by:

$$\psi = xy$$

(i) let $\vec{q} = u\hat{i} + v\hat{j}$ be velocity

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

$$u = -x \quad v = y$$

$$\nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix} = 0$$

$$\therefore \nabla \times \vec{q} = 0 \Rightarrow \text{flow is irrotational}$$

(ii) We know that

$$\frac{\partial \phi}{\partial x} = + \frac{\partial \psi}{\partial y} \quad \& \quad \frac{\partial \phi}{\partial y} = - \frac{\partial \psi}{\partial x}$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = x$$

$$\frac{\partial \phi}{\partial y} = -y$$

on integrating -

$$\phi = \frac{x^2}{2} + f_1(y) \quad \phi = -\frac{y^2}{2} + f_2(x)$$

$\therefore$ overall velocity potential will be

$$\boxed{\phi = \left(\frac{x^2 - y^2}{2} \right) + C}$$

(iii) $\psi = xy$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

$$= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x)$$

$$= 0$$

$$\Rightarrow \boxed{\psi \text{ satisfy laplace equation}}$$

$$\phi = \frac{x^2}{2} - \frac{y^2}{2}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{2}\right) + \frac{\partial}{\partial y}\left(-\frac{y}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

$$\therefore \boxed{\phi \text{ satisfy laplace equation}}$$

(iv) Streamlines are given by $\psi = c$

i.e. $\boxed{xy = c}$

& potential lines are given by $\phi = c$

$$\boxed{x^2 - y^2 = c'}$$

