

NEXT IAS

(To be filled by candidate)

T-4/36

Name of Candidate : ANNU KOUSIK NARASIMHA

Roll No. : OACMTTA 003 Date of Examination : 14/07/2024

Mobile Number : E-mail :

Exam Centre : Old Rajinder Nagar ☒

Online ☐

CSE (MAINS) TEST SERIES-2024

Test-04

MATHEMATICS
OPTIONAL

Dated : 14-07-2024

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are **EIGHT** questions divided in **Two** sections and printed in **ENGLISH**.

Candidate has to attempt **FIVE** questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	20			
2. (b)	18	15			
2. (c)	21	15			
3. (a)	24	20			
3. (b)	27	15			
3. (c)	30	15			
4. (a)	34	15			
4. (b)	37	20			
4. (c)	40	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	15			
6. (b)	57	15			
6. (c)	60	20			
7. (a)	64	15			
7. (b)	67	15			
7. (c)	70	20			
8. (a)	74	20			
8. (b)	78	15			
8. (c)	81	15			
Grand Total					

Remarks

Observations:

SECTION-A

- Q.1 (a) Formulate PDE for surfaces whose tangent planes form a tetrahedron of constant volume with coordinate planes. (10)

The surfaces ^{Tangent planes} are given by the eqn

$$\frac{\partial f}{\partial x}(x-x) + \frac{\partial f}{\partial y}(y-y) = z-z$$

Finding the point of intersection of the planes with the axes:

X-axis $\Rightarrow y = z = 0$

$$\Rightarrow \frac{\partial f}{\partial x}(x-x) + \frac{\partial f}{\partial y}(-y) = -z$$

$$\Rightarrow p(x-x) = yq - z \Rightarrow x = x + \frac{1}{p}(yq - z) \quad \text{--- (1)}$$

$$\text{|| by } y = y + \frac{1}{q}(xp - z) \quad \text{--- (2)} \quad (p = \frac{\partial f}{\partial x}, q = \frac{\partial f}{\partial y})$$

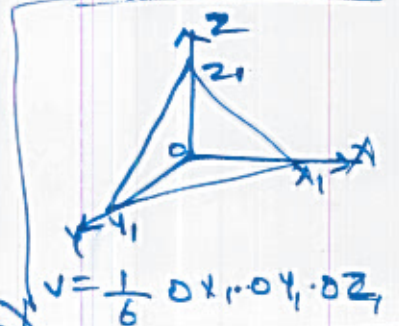
$$\text{and } z = xp + yq \quad \text{--- (3)}$$

Now the volume of tetrahedron is

given by $V = \frac{1}{6} (x_1 \cdot y_2 \cdot z_3) = \text{const}(k)$

$$\Rightarrow 6k = \left(x + \frac{1}{p}(yq - z)\right) \left(y + \frac{1}{q}(xp - z)\right) (xp - yq)$$

$$\Rightarrow 6k = \frac{(px + qy - z)(px + qy - z)}{pq} (px - yq)$$



$$\Rightarrow \delta K = K'(ay) = \frac{(px+qy-z)^2 (px-qy)}{pq}$$

$$\Rightarrow \frac{(px+qy-z)^2 (px-qy) - K'pq}{pq} = 0$$

is the second PDE of the
surfaces



Q.1 (b) State the sufficient condition for convergence of the Gauss Seidel iteration method. (10)

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Considering the system of eqns

$$AX = B$$

$$\text{i.e. } \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

The sufficient condition for the convergence of Gauss seidel method:

$$\text{Now } a_1 x_1 + b_1 y_1 + c_1 z_1 = d_1 \quad \text{--- (1)}$$

$$a_2 x_1 + b_2 y_1 + c_2 z_1 = d_2 \quad \text{--- (2)}$$

$$a_3 x_1 + b_3 y_1 + c_3 z_1 = d_3 \quad \text{--- (3)}$$

$$\Rightarrow x_1 = \frac{1}{a_1} (d_1 - b_1 y_1 - c_1 z_1) \quad \text{From (1).}$$

$$y_1 = \frac{1}{b_2} (d_2 - a_2 x_1 - c_2 z_1) \quad \text{From (2)}$$

$$z_1 = \frac{1}{c_3} (d_3 - a_3 x_1 - b_3 y_1) \quad \text{From (3).}$$

The eqns converge to (x, y, z) if the system is well conditioned.

i.e. ~~sum~~ absolute value of the ^{diagonal} elements of ~~diagonal~~ matrix A is more than sum of absolute values of other elements of the corresponding row.

$$\begin{aligned} \text{i.e. } & |a_1| \geq |b_1| + |c_1| \quad ; \\ & |b_2| \geq |a_2| + |c_2| \quad ; \\ & |c_3| \geq |a_3| + |b_3| \quad ; \end{aligned} \quad \left. \vphantom{\begin{aligned} & |a_1| \geq |b_1| + |c_1| \quad ; \\ & |b_2| \geq |a_2| + |c_2| \quad ; \\ & |c_3| \geq |a_3| + |b_3| \quad ; \end{aligned}} \right\}$$

is the reqd sufficient condition
for convergence.



- Q.1 (c) Express the Boolean function $f(x, y, z) = x + \overline{xy} + \overline{xz} + z$ in disjunctive normal form (DNF) and construct the truth table for the function. (10)

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By Demorgan theorem, $\overline{A+B} = \overline{A} \cdot \overline{B}$

$$\Rightarrow x + \overline{xy} + \overline{xz} + z = x + (\overline{xy}) \cdot (\overline{xz}) + z$$

$$= x + xy \cdot x\overline{z} + z$$

$(\because \overline{\overline{A}} = A)$
 $(\because \overline{A \cdot B} = \overline{A} \cdot \overline{B})$

$$\cancel{= x + x(y \cdot \overline{z}) + z}$$

$$= x(1 + y\overline{z}) + z = \underline{\underline{x + z}}$$

Constructing truth table

x	y	z	$x + z$ (or) $x + \overline{xy} + \overline{xz} + z$
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1
0	1	0	0
0	0	1	1
0	1	1	1
0	0	0	0

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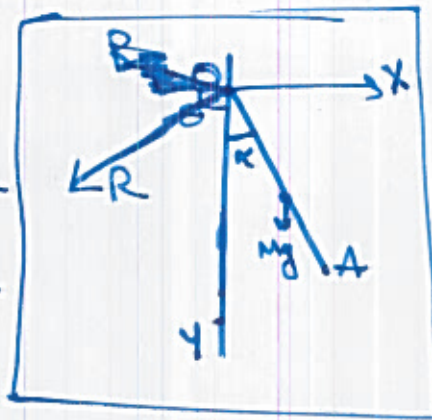


- Q.1 (d) A rod of length $2a$ revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α . Prove that the direction of reaction at the hinge makes with the vertical, an angle

$$\tan^{-1} \left[\frac{3}{4} \tan \alpha \right].$$

(10)

Let the reaction force R make an angle θ with OY .



Given that $\angle NOA = \alpha$

By de Alembert's principle, the sum of reverse effective forces & impressed forces on the system is zero.

In y -direction: Rev eff force = $m \times a_y$
 $= m \times 0 = 0$ — (1)

Impressed forces: $R \cos \theta - Mg$ — (2)

$$\Rightarrow \text{From (1) \& (2), } R \cos \theta - Mg = 0$$

$$\Rightarrow \boxed{R \cos \theta = Mg} \text{ — (A)}$$

In x -direction: Rev eff force acts in OX direction.

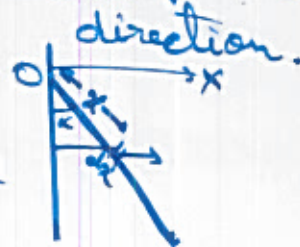
Force on element dx

$$df = dm \times r \times \omega^2$$

$$\Rightarrow df = \frac{M}{2a} (dx) \times x \sin \alpha \omega^2$$

$$\Rightarrow f = \int df = \frac{M \sin \alpha \omega^2}{2a} \int_0^{2a} x dx$$

$$= Ma (\sin \alpha) \omega^2 \text{ — (i)}$$



Impressed force on the system in
x direction = $R \sin \theta$ — (2).

By De-Alembert principle, (1) = (2).

$$\Rightarrow R \sin \theta = M a \omega^2 \sin \alpha \text{ — (3)}.$$

$$\text{Now } \frac{(3)}{(1)} \Rightarrow \frac{R \sin \theta}{R \cos \theta} = \frac{M a \omega^2 \sin \alpha}{M g}$$

$$\Rightarrow \boxed{\tan \theta = \frac{a \omega^2}{g} \sin \alpha} \text{ — (4)}$$

We know that for a revolving rod making
 α angle with vertical, $\boxed{\cos \alpha = \frac{3g}{4a \omega^2}}$

$$\Rightarrow \text{From (4), } \tan \theta = \frac{a \omega^2}{g} \times \frac{\sin \alpha}{\cos \alpha} \times \cos \alpha$$

$$\Rightarrow \tan \theta = \frac{a \omega^2}{g} \times \tan \alpha \times \frac{3g}{4a \omega^2}$$

$$\Rightarrow \tan \theta = \frac{3}{4} \tan \alpha$$

$$\Rightarrow \boxed{\theta = \tan^{-1} \left(\frac{3}{4} \tan \alpha \right)} \text{ is the reqd}$$

angle of reaction made at hinge
with vertical.

Q.1 (e) Find the equation of the streamlines for the flow $q = -(3y^2)\hat{i} - 6x\hat{j}$ at the point $(1, 1)$. (10)

The equation of flow of streamlines
for $q = u\hat{i} + v\hat{j} + w\hat{k}$ is

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{--- (1)}$$

Given that $q = -3y^2\hat{i} - 6x\hat{j}$

$$\Rightarrow u = -3y^2, v = -6x$$

$\Rightarrow$ DE of streamlines is

$$\frac{dx}{-3y^2} = \frac{dy}{-6x} = \frac{dz}{0} \quad \text{--- (2)}$$

$$\Rightarrow \text{From (2), } dz = 0 \Rightarrow \boxed{z = C_1} \quad \text{--- (3)}$$

Taking first two fractions from (2),

$$\frac{dx}{-3y^2} = \frac{dy}{-6x} \Rightarrow \frac{dx}{y^2} = \frac{dy}{2x}$$

$$\Rightarrow 2x dx - y^2 dy = 0 \quad \text{Integrating,}$$

$$\int 2x dx - \int y^2 dy = \int 0$$

$$\Rightarrow x^2 - \frac{y^3}{3} = C_2$$

$$\Rightarrow 3x^2 - y^3 = C_3 \quad \text{At } (1, 1),$$

$$\Rightarrow 3(1)^2 - (1)^3 = C_3 \Rightarrow C_3 = 2$$

$$\Rightarrow \boxed{3x^2 - y^3 = 2} \text{--- (4) .}$$

$\therefore$ The reqd eqn of streamlines are

$$\boxed{3x^2 - y^3 = 2 \text{ and } z = k}$$



Q.2 (a) Find the general solution and singular solution of $6yz - 6pxy - 3qy^2 + pq = 0$.

(20)

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Writing the Charpit eqns for the PDE,
 $f(x, y, z, p, q) = 0$:

$$\frac{dp}{b_x + pf_x} = \frac{dq}{b_y + qf_y} = \frac{dz}{-pf_x - qf_y} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$b_x = -6py, \quad b_y = 6z - 6px - 6qy,$$

$$b_z = 6y, \quad f_p = -6xy + q,$$

$$f_q = -3y^2 + p, \quad b_x + pf_x = 0$$

$$b_y + qf_y = 6z - 6px$$

$$\Rightarrow \frac{dp}{0} = \frac{dq}{6z - 6px} = \frac{dz}{6px + q - 2pq + 3y^2} = \frac{dx}{q - 6xy} = \frac{dy}{3y^2 - p}$$

Taking the first fraction, $dp = 0 \Rightarrow p = C_1$.
 Substituting $p = C_1$ in the PDE, we get

$$6yz - 6C_1xy - 3qy^2 + C_1q = 0$$

$$\Rightarrow 6yz - 6C_1xy = q(3y^2 - C_1)$$

$$\Rightarrow q = \frac{6yz - 6C_1xy}{3y^2 - C_1}$$

$$dz = p dx + q dy$$

$$\Rightarrow dz = C_1 dx + \frac{6yz dy}{3y^2 - C_1} - \frac{6C_1xy dy}{3y^2 - C_1}$$

$$\Rightarrow z = \int c_1 dx + \int \frac{6y^2 dy}{3y^2 - c_1} - \int \frac{6c_1 xy dy}{3y^2 - c_1}$$

$$\Rightarrow z = c_1 x + z \ln(3y^2 - c_1) - x c_1 \ln(3y^2 - c_1)$$

$$\Rightarrow z = x c_1 + \left[\ln(3y^2 - c_1) \right] (z - x c_1)$$

is the reqd singular sol.



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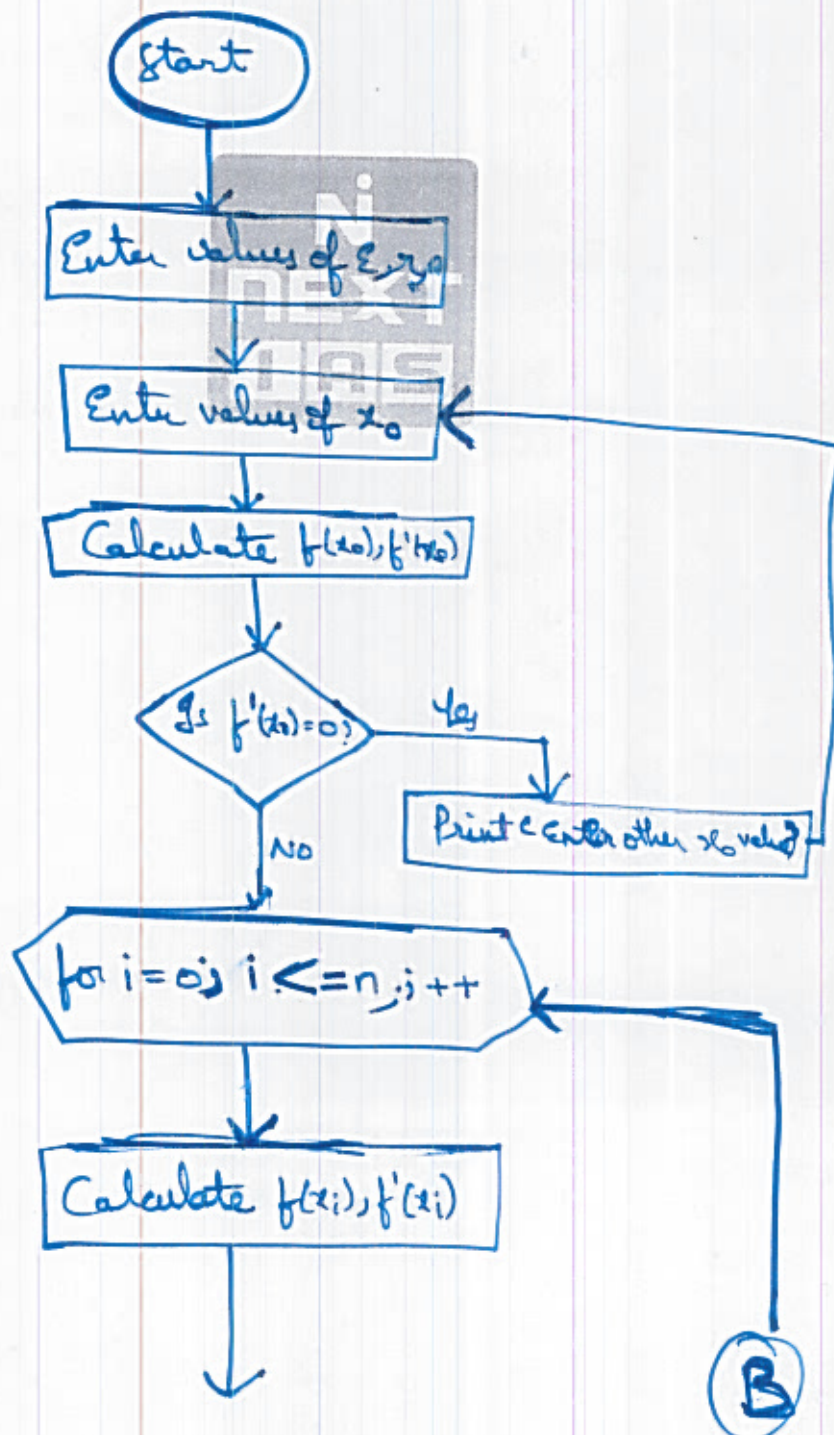
Q.2 (b)

Write an algorithm in the form of a flow chart for Newton-Raphson method. Describe the cases of failure of this method. (15)

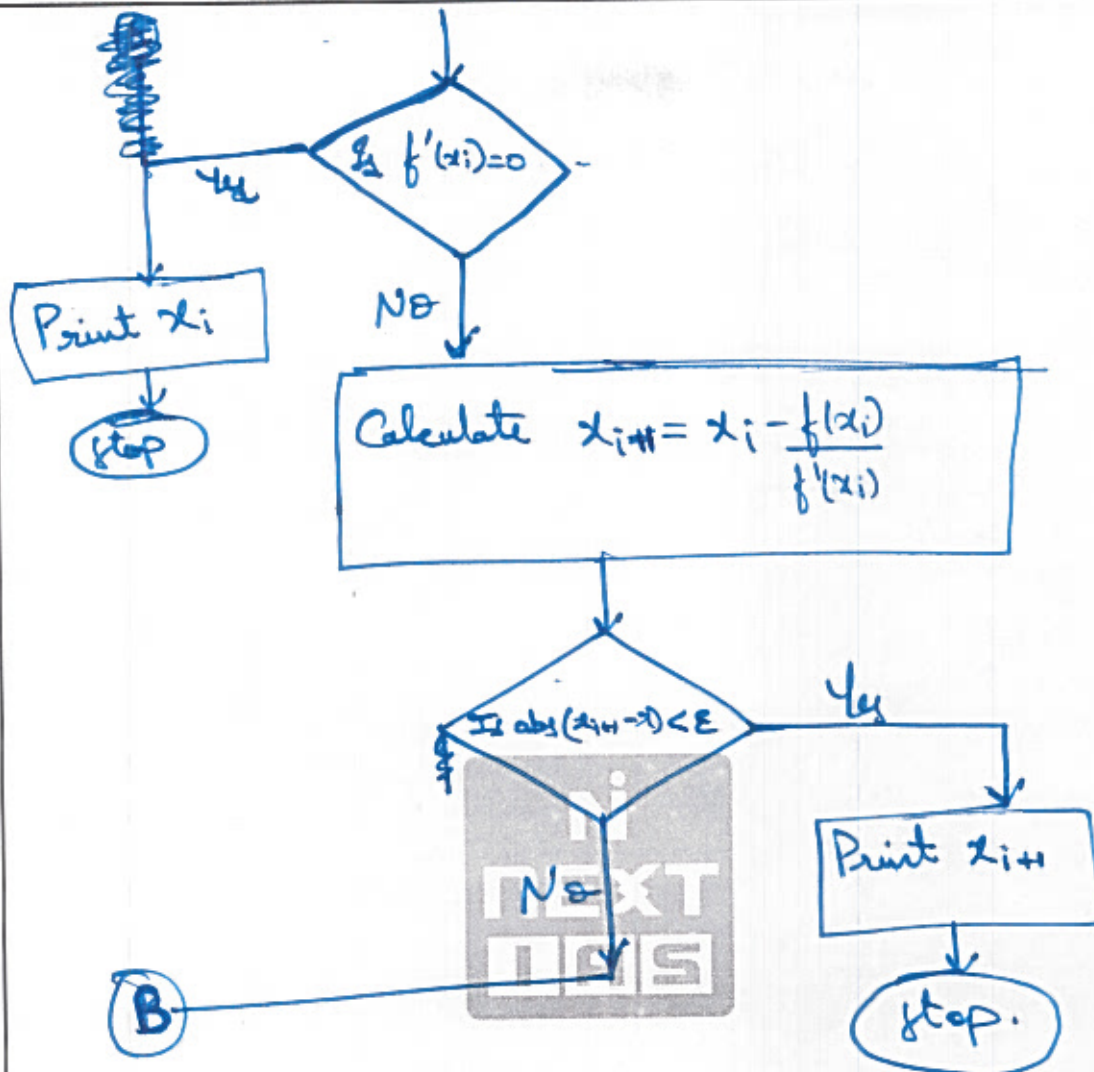
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Newton Raphson method: $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

Let the iteration start with x_0 as initial value, let r be the root and ϵ be the error value, n be no of iterations.



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The cases of failure are :

- (1) for any value of x_i , $f'(x_i) = 0$, then x_{i+1} is undefined
- (2) For $\epsilon_k = |x_k - \alpha|$ and $\epsilon_{k+p} = |x_{k+p} - \alpha|$ where $p > 0$, If $\epsilon_{k+p} > \epsilon_k$ then the method diverges & is a case of failure.

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- Q.2 (c) Show that the surface $\frac{x^2}{a^2 k^2 t^4} + k t^2 \left(\frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = 1$ is a possible form of boundary surface of a liquid at time t . (15)

The surface $F(x, y, z, t)$ is a possible form of boundary if $\frac{DF}{Dt} = 0$ i.e. $\left(\frac{d}{dt} + \mathbf{q} \cdot \nabla \right) F = 0$ where $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$.

$$\frac{d}{dt}(F) = \frac{d}{dt} \left(\frac{x^2}{a^2 k^2 t^4} + k t^2 \left(\frac{y^2}{b^2} + \frac{z^2}{c^2} \right) \right)$$

$$= \frac{-4x^2}{a^2 k^2 t^5} + 2kt \left(\frac{y^2}{b^2} + \frac{z^2}{c^2} \right)$$

Calculating $\mathbf{q} \cdot \nabla F$

$$\nabla F = \mathbf{i} \frac{\partial F}{\partial x} + \mathbf{j} \frac{\partial F}{\partial y} + \mathbf{k} \frac{\partial F}{\partial z}$$

$$= \frac{2x}{a^2 k^2 t^4} \mathbf{i} + \frac{2y k t^2}{b^2} \mathbf{j} + \frac{2z k t^2}{c^2} \mathbf{k}$$

$$\mathbf{q} \cdot \nabla F = 2(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \cdot \left(\frac{2x}{a^2 k^2 t^4} \mathbf{i} + \frac{2y k t^2}{b^2} \mathbf{j} + \frac{2z k t^2}{c^2} \mathbf{k} \right)$$

$$= \frac{2ux}{a^2 k^2 t^4} + \frac{2v k t^2}{b^2} + \frac{2w k t^2}{c^2}$$

$$\frac{dF}{dt} + \mathbf{q} \cdot \nabla F = \frac{-4x^2}{a^2 k^2 t^5} + \frac{2k t y^2}{b^2} + \frac{2k t z^2}{c^2} + \frac{2ux}{a^2 k^2 t^4} + \frac{2v k t^2}{b^2} + \frac{2w k t^2}{c^2}$$

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$$= \frac{-4x^2}{a^2k^2+t^5} + \frac{2kt^2}{b^2} + \frac{2kt^2}{c^2} + \frac{2ux}{a^2k^4+t^4} + \frac{2\cancel{v}kt^2}{b^2} + \frac{2\cancel{w}kb^2}{c^2}$$

$$= \frac{-4x^2}{a^2k^2+t^5} + \frac{2ux}{a^2k^2+t^4} + 2k \left(\frac{t^2y^2}{b^2} + \frac{t^2}{c^2} \right) + 2k \left(\frac{t^2v}{b^2} + \frac{t^2w}{c^2} \right)$$

$$= \frac{-4x^2}{a^2k^2+t^5} + \frac{2ux}{a^2k^2+t^4} + 2kt \left(\frac{y^2}{b^2} + \frac{t^2}{c^2} \right) + 2kt^2 \left(\frac{v}{b^2} + \frac{w}{c^2} \right)$$

$$= \frac{-4x^2}{a^2k^2+t^5} + \frac{2ux}{a^2k^2+t^4} + 2 \left(\frac{1}{t} - \frac{x^2}{a^2k^2+t^5} \right) + 2kt^2 \left(\frac{v}{b^2} + \frac{w}{c^2} \right)$$

(From Initial eqn)

$$= \frac{-6x^2}{a^2k^2+t^5} + \frac{2}{t} + \frac{2ux}{a^2k^2+t^4} + \frac{2kt^2}{b^2} + \frac{2kwet^2}{c^2}$$

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Q.3 (a) Prove that for the equation $z + px + qy - 1 - pqx^2y^2 = 0$, the characteristic strips are given

by $x(t) = \frac{1}{B + Ce^{-t}}$, $y(t) = \frac{1}{A + De^{-t}}$, $z(t) = E - (AC - BD)e^{-t}$, $p(t) = A(B + Ce^{-t})^2$,

$q(t) = B(A + De^{-t})^2$ where A, B, C, D and E are arbitrary constants. Hence find the values of these arbitrary constants if the integral surface passes through the line $z = 0, x = y$.

(20)



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Q.3 (b)

A liquid is contained between two parallel planes, the free surface is circular cylinder of radius ' a ' whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius ' b ' is suddenly annihilated; prove that if Π be the pressure at the outer surface, the initial pressure at any point on the liquid distant ' r ' from the center is

$$\Pi \left(\frac{\log r - \log b}{\log a - \log b} \right). \quad (15)$$

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Q.3 (c)

State the conditions under which Euler's equations of motion can be integrated. Show that

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = F(t) \text{ where the symbols have usual meaning.} \quad (15)$$

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- Q.4 (a) Find a surface satisfying $r + s = 0$ and touching the elliptic paraboloid $z = 4x^2 + y^2$ along its section by the plane $y = 2x + 1$. (15)

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Q.4 (b)

A uniform rod is placed with one end in contact with a horizontal table and is then inclined at an angle of α to the horizon and is allowed to fall. When it becomes horizontal, show

that the angular velocity is $\sqrt{\frac{3g}{2a}} \sin \alpha$ whether the plane be perfectly smooth or perfectly rough. Show that the end of the rod will not leave the plane in either case. (20)

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- Q.4 (c) An infinite mass of fluid is acted on by a force $\frac{\mu}{r^{3/2}}$ per unit mass directed to the origin. If initially, the fluid is at rest and there is cavity in the form of the sphere $r = c$ in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}$. (15)



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SECTION-B

- Q.5 (a) Transform the equation $yz_x - xz_y = 0$ into one in polar coordinates and thereby show that the solution of the given equation represents surfaces of revolution. (10)

$$yz_x - xz_y = 0 \quad \text{let } x = r \cos \theta, y = r \sin \theta$$

$$\Rightarrow z_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \left(\frac{\partial r}{\partial x} \right) + \frac{\partial z}{\partial \theta} \left(\frac{\partial \theta}{\partial x} \right)$$

$$= \frac{\partial z}{\partial r} \left(\frac{x}{r} \right) + \frac{\partial z}{\partial \theta} \left(-\frac{y}{r^2} \right)$$

$$= z_r \cos \theta - \frac{1}{r} \sin \theta z_\theta$$

$$z_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r} \left(\frac{\partial r}{\partial y} \right) + \frac{\partial z}{\partial \theta} \left(\frac{\partial \theta}{\partial y} \right)$$

$$= \frac{\partial z}{\partial r} \left(\frac{y}{r} \right) + \frac{\partial z}{\partial \theta} \left(\frac{1}{r^2} \right)$$

$$= z_r \sin \theta + \frac{1}{r^2} z_\theta$$

$$yz_x - xz_y = 0$$

$$\Rightarrow (r \sin \theta) \left(z_r \cos \theta - \frac{1}{r} \sin \theta z_\theta \right) - r \cos \theta \left(z_r \sin \theta + \frac{1}{r^2} z_\theta \right) = 0$$

~~is the same as before.~~

$$\Rightarrow r z_r \sin \theta \cos \theta - \sin^2 \theta z_\theta - r z_r \cos \theta \sin \theta + \frac{1}{r} \cos \theta z_\theta = 0$$

$$\Rightarrow z_\theta \left(\frac{1}{r} - \sin^2 \theta \right) = 0$$

$$\Rightarrow \frac{\partial z}{\partial \theta} = 0 \Rightarrow \boxed{z = f(r)} \text{ which is a surface of revolution.}$$

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- Q.5 (b) Verify that the equation $yz(y+z)dx + xz(x+z)dy + xy(x+y)dz = 0$ is integrable and find its solution. (10)

Let $\vec{P} = yz(y+z)\mathbf{i} + xz(x+z)\mathbf{j} + xy(x+y)\mathbf{k}$

The eqn is integrable if $\vec{P} \cdot \text{curl} \vec{P} = 0$.

$\text{Curl } \vec{P} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz(y+z) & xz(x+z) & xy(x+y) \end{vmatrix}$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z + yz^2 & x^2z + xz^2 & x^2y + xy^2 \end{vmatrix} = \mathbf{i}(x^2 + 2xy - x^2 - 2xz) \\ - \mathbf{j}(2xy + y^2 - y^2 - 2yz) \\ + \mathbf{k}(2xz + z^2 - 2yz - z^2)$

$= \mathbf{i}(2xy - 2xz) + \mathbf{j}(2yz - 2xy) + \mathbf{k}(2xz - 2yz)$

$\vec{P} \cdot (\nabla \times \vec{P})$

$= yz(y+z)(2x)(y-z) + xz(x+z)(2y)(z-x) \\ + xy(x+y)(2z)(x-y)$

$= 2xyz(y^2 - z^2) + 2xyz(z^2 - x^2) + 2xyz(x^2 - y^2)$

$\Rightarrow \vec{P} \cdot (\nabla \times \vec{P}) = 2xyz(y^2 - z^2 + z^2 - x^2 + x^2 - y^2) \\ = 2xyz(0) = \underline{\underline{0}}$

Hence the equation is integrable.

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Q.5 (c) Express the Boolean function $F(x, y, z) = xy + x'z$ in a product of maxterm form. (10)

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$$\text{Given } F(x, y, z) = xy + x'z$$

$$xy + x'z = (xy + x')(xy + z) \quad (\because AB + CD = (A+B)(A+D))$$

$$= (x + x')(y + x')(x + z)(y + z)$$

$$\cancel{= (x' + y)(x + z)(y + z)} = (x' + y)(x + z)(y + z) \quad (\because A + A' = 1)$$

$$\text{Now } x' + y = x' + y + zz' = (x' + y + z)(x' + y + z')$$

$$\text{Similarly } (x + z) = (x + y + z)(x + y' + z) \text{ and}$$

$$(y + z) = (x + y + z)(x + y + z)$$

$$\therefore F = (x' + y)(x + z)(y + z)$$

$$= (x' + y + z)(x' + y + z')(x + y + z)(x + y' + z) \\ (x + y + z)(x' + y + z)$$

Removing the repetitive terms, ($\because A \cdot A = A$)

$$\therefore F = (x' + y + z)(x' + y + z')(x + y' + z)(x + y + z)$$

is the reqd product of maxterm form.

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Q.5 (d) Convert $(3798.3875)_{10}$ into octal and hexadecimal equivalents.

(10)

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Octal form:

Finding Integer form:

8	3798
8	474 6
8	59 2
8	7 3
0	7

$\therefore 3798 = (7326)_8$

Decimal part

$$0.3875 = 3 \times 8^{-1} + (0.1) 8^{-1} = 3 \times 8^{-1} + 0.0125$$

$$0.0125 = 0 \times 8^{-2} + 0.0125$$

$$0.0125 = 6 \times 8^{-3} + (0.4) 8^{-3} = 6 \times 8^{-3} + 7.8125 \times 10^{-4}$$

$$0.4 \times 8^{-3} = 3 \times 8^{-4} + (0.2) 8^{-4}$$

$$(0.2) 8^{-4} = 1 \times 8^{-5} + (0.6) 8^{-5}$$

$$(0.6) 8^{-5} = 4 \times 8^{-6} + (0.8) 8^{-6} \dots$$

$$\therefore \text{decimal part} = (0.306314\dots)_8$$

$$\therefore (3798.3875)_{10} = (7326.306314\dots)_8$$

Hexadecimal form

Integer part:

16	3798
16	237 6
16	14 D (13)
0	E (14)

$$\therefore (3798)_{10} = \text{ED6}_{16}$$

Decimal part

$$0.3875 = 6 \times 16^{-1} + 0.2 \times 16^{-1}$$

$$(0.2) 16^{-1} = 3 \times 16^{-2} + 0.2 \times 16^{-2}$$

$$(0.2)_{16} = 3 \times 16^{-3} + 0.2 \times 16^{-3}$$

$$(0.2)_{16} = 3 \times 16^{-4} + (0.2)_{16}$$

$$(0.2)_{16} = 3 \times 16^{-5} + (0.2)_{16}$$

!

$$\therefore \text{decimal part} = (0.6333\ldots)_{16}$$

$$\therefore (3798.3875)_{10} = (ED6.6333\ldots)_{16}$$

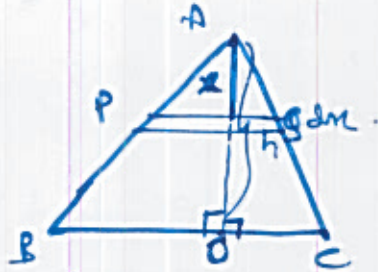


Q.5 (e) Find the moment of inertia of triangular lamina.

(10)

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Consider a triangular lamina ^{ABC} of sides a, b, c and h be the $\perp$ height from A onto BC .



Finding MoI through A ~~in the plane~~ in the plane ABC (Assuming this particular axis as the question does not mention any axis or point through which MoI is to be calculated) parallel to side BC .

In ΔPQ & ΔBC (similar Δ s), $\frac{x}{h} = \frac{PQ}{BC}$
 $\Rightarrow PQ = \frac{xa}{h}$. Mass of the strip PQ is

$$dm = \int PQ dx = \frac{xa}{h} dx (\int)$$

$$dI = dm x^2 = \int \frac{xa}{h} dx x^2$$

$$\Rightarrow I = \int dI = \int_0^h \frac{xa}{h} x^2 dx$$

$$= \frac{a}{h} \times \left(\frac{x^4}{4} \right)_0^h = \frac{ah^3}{4}$$

Let the Mass of lamina be M .

$$\text{Then } \rho = \frac{M}{\Delta ABC} = \frac{M}{\frac{1}{2}ah} \Rightarrow \rho ah = 2M$$

$$\Rightarrow I = \frac{2Mh^2}{4} = \frac{Mh^2}{2}$$

~~Now, $AO = h$ in the Δ AOB & AOC ,~~

~~$$\Delta^k AOB: AB^2 = AO^2 + OB^2 \Rightarrow b^2 = h^2 + OB^2$$~~

~~$$\text{In } \Delta AOC: AC^2 = AO^2 + OC^2 \Rightarrow b^2 = h^2 + OC^2$$~~

~~$$\Rightarrow OB = \sqrt{c^2 - h^2}, \quad OC = \sqrt{b^2 - h^2}$$~~

~~$$\text{Now } OB + OC = BOC = a$$~~

~~$$\Rightarrow a = \sqrt{c^2 - h^2} + \sqrt{b^2 - h^2}$$~~

$$\text{Now } h = AB \sin \angle ABO = AB \sin B = c \sin B$$

$$\Rightarrow h = c \sqrt{1 - \cos^2 B} \quad \cos^2 B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\Rightarrow h^2 = c^2 - c^2 \cos^2 B = c^2 - \frac{c}{2a} (a^2 + c^2 - b^2)$$

~~$$\Rightarrow I = \frac{M}{2} \left(\frac{c}{2a} \right) (a^2)$$~~

$$\Rightarrow I = \frac{M}{2} \left[c^2 - \frac{c}{2a} (a^2 + c^2 - b^2) \right]$$

is the reqd $M.I.$

Q.6 (a) Solve: $r+s-2t=(2x+y)^{1/2}$.

(15)

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$$\Rightarrow D^2 + DD' - 2D'^2 = (2x+y)^{1/2}.$$

Finding complementary fn z_c :

$$\Rightarrow \left(\frac{D}{D'}\right)^2 + \left(\frac{D}{D'}\right) - 2 = 0 \Rightarrow m^2 + m - 2 = 0$$

$$(m = 0/1)$$

$$\Rightarrow (m+2)(m-1) = 0 \Rightarrow \underline{m = -2, m = 1}$$

$$\therefore z_c = \phi_1(y-2x) + \phi_2(y+x) \quad \text{--- (1)}$$

where ϕ_1, ϕ_2 are arbitrary fns. ($\because z_c$ for $m = a$ is $\phi(ax+y)$)Finding particular integral z_p :

$$\Rightarrow z_p = \frac{1}{D^2 + DD' - D'^2} (2x+y)^{1/2}$$

$$\Rightarrow z_p = \frac{1}{(D+2D')(D-D')} (2x+y)^{1/2}$$

we know that $\frac{1}{f(D,D')} g(ax+by)$

$$= \frac{1}{f(h,b)} \iint g(v) \, du \, dv$$

where $v = ax+by$

$$\Rightarrow z_p = \frac{1}{(2+2(v))(2-1)} \iint v^{1/2} \, du \, dv$$

$$(v = 2x+y)$$

$$\Rightarrow z_p = \frac{1}{4} \iint v^{1/2} \, du \, dv$$

$$\Rightarrow Z_p = \frac{1}{4} \int \left(\frac{v^{1/2+1}}{\frac{1}{2}+1} \right) dv = \frac{1}{4} \times \frac{2}{3} \int v^{3/2} dv$$

$$\Rightarrow Z_p = \frac{1}{6} \int v^{3/2} dv = \frac{1}{6} \left(\frac{v^{3/2+1}}{3/2+1} \right)$$

$$\Rightarrow Z_p = \frac{1}{6} \times \frac{2}{5} v^{5/2} = \frac{1}{15} v^{5/2}$$

$$\Rightarrow Z_p = \frac{1}{15} (2x+y)^{5/2} \text{ --- (2).}$$

From (1) & (2), $Z = Z_c + Z_p$

$$\Rightarrow Z = \phi_1(y-2x) + \phi_2(y+x) + \frac{1}{15} (2x+y)^{5/2}$$

is the reqd soln.

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Q.6 (b) Use Gauss Quadrature formula of six points to evaluate $\int_0^1 \frac{dx}{1+x^2}$, given

$$x_1 = -0.23861919, w_1 = 0.46791393$$

$$x_2 = -0.66120939, w_2 = 0.36076157$$

$$x_3 = -0.93246951, w_3 = 0.17132449$$

(15)

$$x_4 = -x_1, x_5 = -x_2, x_6 = -x_3, w_4 = w_1, w_5 = w_2, \text{ and } w_6 = w_3.$$



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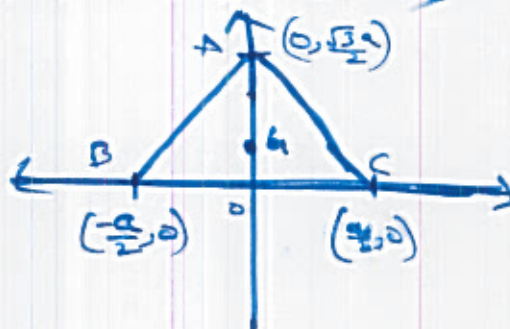


Q.6 (c)

Three parallel rectilinear vortices of the same strength k and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a . Show that all the vortices move round the same cylinder with uniform speed in time $(4\pi^2 a^2)/3k$. (20)

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Considering three vortices of strength k along the sides of equilateral Δk around the origin, at the points $(-\frac{a}{2}, 0)$, $(\frac{a}{2}, 0)$, $(0, \frac{\sqrt{3}a}{2})$.



The complex potential due to the vortex of strength k is $w = \frac{ik}{2\pi} \ln(z - z_1)$.

Considering the point A, the complex potential is $w_A = \frac{ik}{2\pi} \left(\ln(z - (-\frac{a}{2})) + \ln(z - (\frac{a}{2})) \right)$

$$\Rightarrow w_A = \frac{ik}{2\pi} \ln\left(z^2 - \frac{a^2}{4}\right) \quad \text{Let } z = x + iy.$$

$$\text{Then } z^2 = x^2 - y^2 + 2ixy$$

$$\Rightarrow w_A = \frac{ik}{2\pi} \ln\left(\left(x^2 - y^2 - \frac{a^2}{4}\right) + 2ixy\right)$$

$$\text{Let } \left(x^2 - y^2 - \frac{a^2}{4} + 2ixy\right) = R e^{i\theta}, \text{ then.}$$

$$w_A = \frac{ik}{2\pi} (\ln R + i\theta) = \phi + i\psi \quad (\text{say})$$

$$\text{Then } \psi = \frac{k}{4\pi} \ln R_1 = \frac{k}{4\pi} \ln \left(\sqrt{(x^2 - y^2 - \frac{a^2}{4})^2 + (2xy)^2} \right)$$

$$= \frac{k}{4\pi} \ln \left((x^2 - y^2 - \frac{a^2}{4})^2 + 4x^2y^2 \right) \quad \text{--- (1)}$$

Now $\frac{\partial \psi}{\partial r} = 0$ from (1), then there is

~~no~~ no velocity component outward.

$\therefore$ They move circular.

$$\text{Now, } v_r = \frac{\partial \psi}{\partial r}$$

At A, $x=0, y = \frac{\sqrt{3}}{2}a$. Now Calculating

$$\left. \frac{\partial \psi}{\partial y} \right|_{(x,y) = (0, \frac{\sqrt{3}}{2}a)}$$

$$\frac{\partial \psi_A}{\partial y} = \frac{\partial}{\partial y} \left[\frac{k}{4\pi} \ln \left((x^2 - y^2 - \frac{a^2}{4})^2 + 4x^2y^2 \right) \right] \Bigg|_{(0, \frac{\sqrt{3}}{2}a)}$$

$$= \frac{k}{4\pi} \times \frac{1 \times 2(x^2 - y^2 - \frac{a^2}{4})(-2y) + 8x^2y}{[(x^2 - y^2 - \frac{a^2}{4})^2 + 4x^2y^2]}$$

$$= \frac{k}{4\pi} \times \frac{-4 \times \frac{\sqrt{3}a}{2} \times (0^2 - (\frac{\sqrt{3}a}{2})^2 - \frac{a^2}{4}) + 0}{(0^2 - (\frac{\sqrt{3}a}{2})^2 - \frac{a^2}{4})^2 + 0}$$

$$\Rightarrow \frac{\partial \psi_A}{\partial y} = \frac{k}{4\pi} \times \frac{-2\sqrt{3}a}{-\frac{3a^2}{4} - \frac{a^2}{4}}$$

$$\Rightarrow \left. \frac{\partial \psi}{\partial y} \right|_{y = \frac{\sqrt{3}a}{2}} = \frac{K}{4\pi} \times \frac{2\sqrt{3}a}{a^2}$$

$$\Rightarrow \left. \frac{\partial \psi}{\partial y} \right|_{y = \frac{\sqrt{3}a}{2}} = \frac{\sqrt{3}K}{2\pi a} = \text{velocity at A.}$$

Now the cylinder has centre at G
where $AG = \frac{a}{\sqrt{3}}$ (centre of equilateral Δk)

$$\Rightarrow \text{Circumference of cylinder} = 2\pi \left(\frac{a}{\sqrt{3}} \right)$$

$\therefore$ Time taken to cover the cylinder

$$\Rightarrow t = \frac{\text{Circumference}}{\text{speed}}$$

$$\Rightarrow t = \frac{2\pi \frac{a}{\sqrt{3}}}{\frac{\sqrt{3}K}{2\pi a}} = \frac{4\pi^2 a^2}{3K}$$

$\therefore$ The vortices move around the cylinder in time period $\frac{4\pi^2 a^2}{3K}$.
Hence proved.

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- Q.7 (a) Find the family of surfaces orthogonal to the family of surfaces given by the differential equation $(y+z)p + (z+x)q = x+y$. (15)

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Q.7 (b) Using Runge Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$.

Use four decimal places for calculation and step length 0.2. (15)



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Q.7 (c) For a dynamical system having two degrees of freedom, the Lagrangian is given by

$$L = \frac{m}{2} \left(a^2 \dot{q}_1^2 + \dot{q}_2^2 \right) - \frac{k}{2} (a^2 + q_2^2), \text{ where } q_1 \text{ and } q_2 \text{ are generalized coordinates. Find the}$$

corresponding Hamiltonian and derive the Hamiltonian equations of motion. Show further that the generalized momentum corresponding to q_1 is constant. Show that the system exhibits a simple harmonic motion with respect to the generalized coordinate q_2 . (20)

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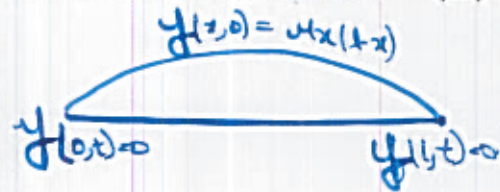
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- Q.8 (a) A tightly stretched string has its ends fixed at $x = 0$ and $x = 1$. At time $t = 0$, the string is given a shape defined by $f(x) = \mu x(1-x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at time $t > 0$. (20)

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The string (vibrating), satisfies the wave eqn



$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ① let $y = X(x)T(t)$ be the solution to the eqn.

$$\frac{\partial^2 y}{\partial t^2} = X T'' \quad , \quad \frac{\partial^2 y}{\partial x^2} = X'' T \quad \text{From ①,}$$

$$X T'' = c^2 X'' T \Rightarrow \frac{T''}{T} = \frac{c^2 X''}{X} \quad \text{②}$$

From ②, let $\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$ (say) ③

Case ①: let $\lambda = 0 \Rightarrow T = C_1 t + C_2$ and

$$X = C_3 x + C_4$$

$\Rightarrow y = X T = (C_1 t + C_2)(C_3 x + C_4)$ does not satisfy the condition, as it is linear, not periodic.

Case ②: let $\lambda = p^2 \Rightarrow T'' - p^2 T = 0$

$$\Rightarrow T = C_1 e^{cpt} + C_2 e^{-cpt}$$

$$X'' - X = 0 \Rightarrow X = C_3 e^x + C_4 e^{-x}$$

$$\Rightarrow y = XT = (C_1 e^{cpt} + C_2 e^{-cpt}) (C_3 e^x + C_4 e^{-x})$$

does not satisfy the condition as the string is periodic, not exponential.

III) Let $\lambda = -p^2$. From ②,

$$\Rightarrow T'' + C^2 p^2 T = 0 \Rightarrow T = C_3 \sin cpt + C_4 \cos cpt$$

$$\Rightarrow X'' + p^2 X = 0 \Rightarrow X = C_1 \sin px + C_2 \cos px$$

$$\Rightarrow y = XT = (C_1 \sin px + C_2 \cos px) (C_3 \sin cpt + C_4 \cos cpt)$$

is the reqd soln to wave eqn, which is periodic.

Given: $y(0, t) = 0$, $y(l, t) = 0$, $y(x, 0) = M \sin(l-x)$

and $\frac{dy}{dt} \Big|_{t=0} = 0$.

(i) $y(0, t) = 0 \Rightarrow 0 = (C_1(0) + C_2(1)) (C_3 \sin cpt + C_4 \cos cpt)$

$\Rightarrow C_2 = 0$ ($\because$ ~~$C_3 \sin cpt + C_4 \cos cpt \neq 0 \forall t$~~)

$\Rightarrow y = (C_1 \sin px) (C_3 \sin cpt + C_4 \cos cpt) \quad \text{--- (8)}$

(ii) Given $\frac{dy}{dt} \Big|_{t=0} = 0$.

$\Rightarrow \frac{dy}{dt} = (C_1 \sin px) (C_3 c \cos cpt - C_4 c \sin cpt) = 0$

$$\left. \frac{dy}{dt} \right|_{t=0} = 0 = (C_1 \sin px) (C_3 4 \cos 0 - C_4 c p \sin 0)$$

$$\Rightarrow \cancel{C_3} C_3 = 0 \quad (\because \text{If } C_1 = 0, y = 0 \forall x) \\ \therefore C_1 \neq 0.$$

$$\therefore y = C_1 C_4 \sin px \cos c p t$$

$$\text{(iii) now } y|_{(l,t)} = 0 = C_1 C_4 \sin pl \cos c p t \\ \Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$$

$$\therefore y = C_1 C_4 \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l}$$

$$\text{Now } y(x,0) = Mx(l-x)$$

$$\Rightarrow y(x,0) = C_1 C_4 \sin \frac{n\pi x}{l} = \sum b_n \sin \frac{n\pi x}{l} \text{ (say)} \\ = Mx(l-x).$$

Using Half range fourier expansion,

$$b_n = \frac{2}{l} \int_0^l Mx(l-x) \sin \frac{n\pi x}{l} dx$$

$$\Rightarrow b_n = \frac{2M}{l} \int_0^l (lx - x^2) \sin \left(\frac{n\pi x}{l} \right) dx$$

$$\int_0^l x \sin \frac{n\pi x}{l} dx = \left(x \times \frac{-1}{n\pi} \cos \frac{n\pi x}{l} \right)_0^l + \int_0^l \frac{1}{n\pi} \cos \frac{n\pi x}{l} dx$$

$$= \left(-\frac{x}{n\pi} \cos \frac{n\pi x}{l} + \frac{1}{n^2 \pi^2} \sin \frac{n\pi x}{l} \right)_0^l$$

$$= -\frac{l^2}{n\pi} (-1)^n = \frac{l^2}{n\pi} (-1)^{n+1}$$

$$\therefore \int_0^l lx \sin \left(\frac{n\pi x}{l} \right) dx = \frac{l^3}{n\pi} (-1)^{n+1}$$

$$\begin{aligned}
 \int_0^l x^2 \frac{\sin n\pi x}{l} &= \left[x^2 \left(-\frac{1}{n\pi} \right) \cos \left(\frac{n\pi x}{l} \right) \right] + \int \frac{2x}{n\pi} \cos \frac{n\pi x}{l} \\
 &= -\frac{l x^2}{n\pi} \cos \frac{n\pi x}{l} + \frac{2l}{n\pi} \int x \cos \frac{n\pi x}{l} \\
 &= -\frac{l x^2}{n\pi} \cos \left(\frac{n\pi x}{l} \right) + \frac{2l}{n\pi} \left[x \times \frac{1}{n\pi} \sin \frac{n\pi x}{l} - \int \frac{1}{n\pi} \sin \frac{n\pi x}{l} \right] \\
 &= \left[-\frac{l x^2}{n\pi} \cos \frac{n\pi x}{l} + \frac{2x l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} + \frac{2l^3}{n^3 \pi^3} \cos \frac{n\pi x}{l} \right]_0^l
 \end{aligned}$$

$$= -\frac{l^3}{n\pi} \cos(-1)^n + \frac{2l^3}{n^2 \pi^2} (-1)^n - \frac{2l^3}{n^3 \pi^3}$$

$$\Rightarrow b_n = \frac{2\mu}{l} \int_0^l (lx - x^2) \sin \left(\frac{n\pi x}{l} \right) dx$$

$$= \frac{2\mu}{l} \left[\frac{l^3}{n\pi} (-1)^{n+1} - \frac{l^3}{n\pi} (-1)^{n+1} + \frac{2l^3}{n^2 \pi^2} (1 - (-1)^n) \right]$$

$$\text{For } n=2m-1, \quad b_n = \frac{2\mu}{l} \times \frac{4l^3}{n^2 \pi^2} = \frac{8\mu l^2}{n^2 \pi^2}$$

(∵ n ≠ even)
as y ≠ 0

$$\Rightarrow y = \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{l} \cos n\pi \frac{ct}{l}$$

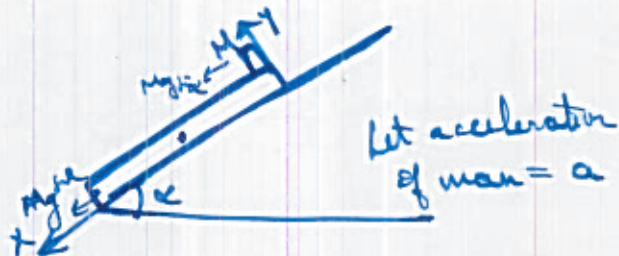
$$\Rightarrow y = \sum_{n=1}^{\infty} \frac{8\mu l^2}{(2m-1)^2 \pi^2} \frac{\sin \frac{(2m-1)\pi x}{l} \cos \frac{(2m-1)\pi ct}{l}}{l}$$

is the reqd soln of wave eqn.

- Q.8 (b) A plank of mass m and length $2a$ is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, a man of mass, M , starting from the upper end walks down the plank so that it does not move,

show that he will reach the other end in time $\sqrt{\frac{4Ma}{(m+M)g \sin \alpha}}$. (15)

Using D'Alembert principle, the sum of impressed force & reverse effective force = 0



Impressed force on the system = $Mg \sin \alpha + mg \sin \alpha$
in the direction X .

Reverse effective force = $-M \times \frac{d^2x}{dt^2} - m \times (0)$
($\because$ plank is const)

$\therefore$ By D'Alembert principle,

$$Mg \sin \alpha + mg \sin \alpha - M \frac{d^2x}{dt^2} = 0.$$

$$\Rightarrow (M+m)g \sin \alpha = M \frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{(M+m)g \sin \alpha}{M} = \frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{dx}{dt} = \int \frac{d^2x}{dt^2} dt = \frac{(M+m)g \sin \alpha}{\mu} t$$

$$\Rightarrow \int_0^{2a} dx = \int_0^t \frac{(M+m)g \sin \alpha}{\mu} t dt$$

$$\Rightarrow 2a = \frac{(M+m)g \sin \alpha}{\mu} \left(\frac{t^2}{2} \right)_0^t$$

$$\Rightarrow \frac{4a\mu}{(M+m)g \sin \alpha} = t^2.$$

$$\Rightarrow t = \sqrt{\frac{4a\mu}{(M+m)g \sin \alpha}}$$

is the
reqd time
for the
man to reach the
other end of plank.

Hence proved.

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Q.8 (c) A two-dimensional flow field is given by $\psi = xy$.

- (i) Show that the flow is irrotational.
- (ii) Find the velocity potential.
- (iii) Verify that ψ and ϕ satisfy the Laplace equation
- (iv) Find the streamlines and potential lines. (15)

(i) Given $\psi = xy$. The flow is irrotational
if $\nabla \times \mathbf{q} = 0$. Now $\mathbf{q} = u\mathbf{i} + v\mathbf{j} = -\frac{\partial \phi}{\partial x}\mathbf{i} - \frac{\partial \phi}{\partial y}\mathbf{j}$
 $\Rightarrow \mathbf{q} = -\frac{\partial \psi}{\partial y}\mathbf{i} + \frac{\partial \psi}{\partial x}\mathbf{j}$ ($\because \phi$ & ψ obey Cauchy Riemann eqns)

$$\Rightarrow \mathbf{q} = -x\mathbf{i} + y\mathbf{j}$$

$$\nabla \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix} = \mathbf{i} \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y) \right) - \mathbf{j} \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-x) \right) + \mathbf{k} \left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(-x) \right)$$

$$= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(0) = \underline{\underline{0}}.$$

$\therefore \nabla \times \mathbf{q} = 0 \Rightarrow$ Flow is irrotational.

(ii) ~~is~~ finding velocity potential ϕ .

Now, ϕ & ψ obey Cauchy Riemann eqns.

$$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \& \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$\hookrightarrow \textcircled{1} \qquad \qquad \qquad \hookrightarrow \textcircled{2}$

From $\textcircled{1}$, $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = x \Rightarrow \phi = \frac{x^2}{2} + f(y)$

From $\textcircled{2}$, $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = -y = f'(y)$

$$\Rightarrow f(y) = -\frac{y^2}{2} + C$$

$$\therefore \phi = \frac{x^2}{2} - \frac{y^2}{2} + C \text{ is the read velocity potential.}$$

(iii) Verifying Laplace eqn:

$$\begin{aligned} \text{For } \psi: \nabla^2 \psi &= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{\partial^2}{\partial x^2}(xy) + \frac{\partial^2}{\partial y^2}(xy) \\ &\quad + \frac{\partial^2}{\partial z^2}(xy) \\ &= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + 0 = \underline{0} = \nabla^2 \psi \end{aligned}$$

$$\begin{aligned} \text{For } \phi: \nabla^2 \phi &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial^2}{\partial x^2}\left(\frac{x^2 - y^2}{2} + C\right) \\ &\quad + \frac{\partial^2}{\partial y^2}\left(\frac{x^2 - y^2}{2} + C\right) + \frac{\partial^2}{\partial z^2}\left(\frac{x^2 - y^2}{2} + C\right) \end{aligned}$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + 0$$

$$= 1 - 1 + 0 = \underline{0} = \nabla^2 \phi$$

$$\therefore \nabla^2 \phi = 0 \text{ and } \nabla^2 \psi = 0. \text{ Hence proved}$$

(iv) Finding streamlines:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{0}$$

Candidates
must not write
on this margin

$$\Rightarrow dz=0 \Rightarrow \boxed{z=c_1}; \quad \frac{dx}{x} = \frac{dy}{y}$$
$$\Rightarrow \int \frac{dx}{x} + \int \frac{dy}{y} = 0 \Rightarrow \ln x + \ln y = c_2$$
$$\Rightarrow \boxed{xy=c_2'}$$

$\therefore$ streamlines are $\boxed{xy=c_2' \text{ and } z=c_1}$



SPACE FOR ROUGH WORK

