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NEXT IAS

(To be filled by candidate)

Name of Candidate : VAIBHAV PANDEY

Roll No. : OAC24MTTN004 Date of Examination : 26/7/24

Mobile Number : ..... E-mail ....

Exam Centre : Old Rajinder Nagar ☐ Online ☑

CSE (MAINS) TEST SERIES-2024

Test-03	MATHEMATICS OPTIONAL
Dated : 07-07-2024	

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

*There are **EIGHT** questions divided in **Two sections** and printed in **ENGLISH**.*

*Candidate has to attempt **FIVE** questions in all.*

***Question No. 1 and 5** are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.*

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	20			
2. (c)	21	15			
3. (a)	24	20			
3. (b)	28	15			
3. (c)	31	15			
4. (a)	34	15			
4. (b)	37	20			
4. (c)	41	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	20			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	81	20			
Grand Total					

Remarks

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Observations:

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SECTION-A

Q.1 (a) Find the singular solution of the differential equation $9p^2(2-y)^2 = 4(3-y)$. **(10)**

$$\left(\frac{dy}{dx}\right)^2 (2-y)^2 = \frac{4}{9}(3-y)$$

$$\Rightarrow \int \frac{2-y}{\sqrt{3-y}}\,dy = \int \pm\frac{2}{3}\,dx$$

$$\Rightarrow \int \sqrt{3-y}\,dy - \int \frac{dy}{\sqrt{3-y}} = \int \pm\frac{2}{3}\,dx$$

$$\Rightarrow \frac{2}{3}(3-y)^{3/2} - 2\sqrt{3-y} = \int \pm\frac{2}{3}\,dx$$

$$\Rightarrow 2\sqrt{3-y}\left(\frac{-y}{3}\right) = \pm\int\frac{2}{3}\,dx$$

$$\Rightarrow \pm y\sqrt{3-y} = x + c$$

{ c is arbitrary constant }

$$\Rightarrow (x+c)^2 - y^2(3-y) = 0$$

is the general solution.

Now, **p - discriminant :-**

$$(2-y)^2(3-y) = 0 \quad \text{---} \quad (1)$$

And, **c - discriminant :-**

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g.s. is $c^2 + 2xc + x^2 - y^2(3-y) = 0$

$\therefore$ c-discriminant is

$$4x^2 - 4\left(x^2 - y^2(3-y)\right) = 0$$

$$\Rightarrow y^2(3-y) = 0 \quad \text{---} \quad (2)$$

From (1) and (2), the common factor is $3-y=0$

$$\Rightarrow y = 3$$

It also satisfies the original D.E.

$\therefore$ singular solution is $\boxed{y = 3}$

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Q.1 (b) Using Laplace Transform, solve the IVP $ty'' + 2ty' + 2y = 2; y(0) = 1$ and $y'(0)$ is arbitrary. Does this problem has a unique solution. **(10)**

$$\mathcal{L}\{y''(t)\} = p^2 \mathcal{L}\{y(t)\} - p\,y(0) - y'(0)$$

$$= p^2 \mathcal{L}\{y(t)\} - p - y'(0)$$

$$\mathcal{L}\{y'(t)\} = p\,\mathcal{L}\{y(t)\} - 1$$

$$\therefore \mathcal{L}\{ty''(t)\} = -\frac{d}{dp}\left(p^2 \mathcal{L}\{y(t)\} - p - y'(0)\right)$$

$$= -\left(2pY(p) + p^2 Y'(p) - 1\right)$$

$$\{\text{Let } \mathcal{L}\{y(t)\} = Y(p)\}$$

$$\text{And } \mathcal{L}\{ty'(t)\} = -\frac{d}{dp}\{pY(p) - 1\}$$

$$= -\left(Y(p) + pY'(p)\right)$$

$$\therefore \mathcal{L}\{ty'' + 2ty' + 2y\}$$

$$= -2pY(p) - p^2 Y'(p) + 1 - 2Y(p) - 2pY'(p) + 2Y(p)$$

$$\text{Also, } \mathcal{L}\{2\} = \frac{2}{p}$$

$$\therefore \text{we have, } -2pY(p) - (p^2 + 2p)Y'(p) = \frac{2}{p} - 1$$

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$$\Rightarrow Y'(p) + \frac{2}{p+2} Y(p) = \frac{1}{p(p+2)} - \frac{2}{p^2(p+2)}$$

$$\text{I.F.} = e^{\int \frac{2}{p+2} dp} = e^{2\ln(p+2)} = (p+2)^2$$

$$\Rightarrow (p+2)^2 Y(p) = \int \left(\frac{p+2}{p} - \frac{2(p+2)}{p^2} \right) dp$$

$$\Rightarrow (p+2)^2 Y(p) = p + 2\ln p + \frac{4}{p} + C$$

{C → arbitrary constant}

$$\Rightarrow Y(p) = \frac{p}{(p+2)^2} + \frac{4}{p(p+2)^2} + \frac{C}{(p+2)^2}$$

$$\Rightarrow y(t) = L^{-1} \left\{ \frac{1}{p+2} - \frac{2}{(p+2)^2} + \frac{1}{p} - \frac{1}{p+2} - \frac{2}{(p+2)^2} + \frac{C}{(p+2)^2} \right\}$$

$$\Rightarrow y(t) = 1 + (C-4) t e^{-2t}$$

$$\Rightarrow \boxed{y(t) = 1 + C' t e^{-2t}}$$

where C' is an arbitrary constant

$$\boxed{\therefore y(t) = 1 + y'(0) t e^{-2t}}$$

$\therefore y(t)$ depends on $y'(0)$ & hence is not unique.

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Q.1 (c) Evaluate $\int_{0,0}^{2,1}(10x^4-2xy^3)dx-3x^2y^2dy$ along the path $x^4-6xy^3=4y^2$. **(10)**

Integral is of the form $\int_C M\,dx+N\,dy$

$$\frac{\partial M}{\partial y}=-6xy^2$$

$$\frac{\partial N}{\partial x}=-6xy^2=\frac{\partial M}{\partial y}$$

$\therefore$ the integrand is exact & hence independent of path.

Let $\frac{\partial \phi}{\partial x}=10x^4-2xy^3$

$\Rightarrow \phi=2x^5-x^2y^3+f(y)$

And $\frac{\partial \phi}{\partial y}=-3x^2y^2$

$\Rightarrow f'(y)=0 \Rightarrow f(y)=c$

$\therefore \phi=2x^5-x^2y^3+c$

$$\therefore I=\int_{(0,0)}^{(2,1)} d\left(10x^4-2xy^3\right)dx-3x^2y^2\,dy$$

$$=\int_{(0,0)}^{(2,1)} d\phi$$

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$= \phi(2,1) - \phi(0,0)$

$= \boxed{60}$



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Q.1 (d) If mass density at any point of a cord varies as the radius of curvature of the curve in which it hangs freely under gravity, then prove that this curve is the catenary of uniform strength. **(10)**



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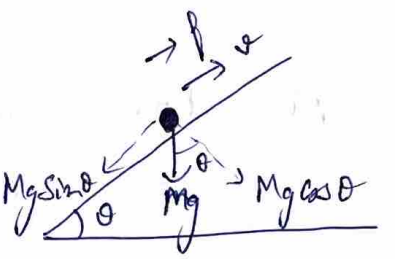
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Q.1 (e) A train of mass M kg is ascending a smooth incline of 1 in n and velocity is v m/s, its acceleration is f m/s^2. Prove that the effective power of the engine is $Mv(nf+g)/n$ watts. **(10)**

Let $t=0$ at the moment velocity is v m/s.

Let $W(t)$ be the total work done by the engine at time t starting from $t=0$



Let the train cover a distance s along the inclined plane in time t and its velocity be V.

$\therefore V^2 = v^2 + 2fs$ — (1) { By $v^2 = u^2 + 2as$ for constant a }

Also, $s = vt + \frac{1}{2}ft^2$ — (2)

Now, by Work Energy Theorem :-

$W(t)$ + work done by gravity = Δ K.E.

$\Rightarrow W(t) + \left(-(Mg\sin\theta)s\right) = \frac{1}{2}MV^2 - \frac{1}{2}Mv^2$

$\Rightarrow W(t) = Mgs\sin\theta + \frac{1}{2}M \cdot 2fs$ { from (1) }

$\Rightarrow W(t) = M(f + g\sin\theta)s$

$\Rightarrow W(t) = M\left(f + \frac{g}{n}\right)\left(vt + \frac{1}{2}ft^2\right)$ { from (2) and $\sin\theta = 1/n$ }

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$\therefore$ Power $= \dfrac{dW(t)}{dt}$

$= \dfrac{M(nf+g)}{n}(v+ft)$

At $t=0$, i.e. under the given conditions

effective power of engine

$= \dfrac{Mv(nf+g)}{n}$ //

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Q.2 (a) Show that the equation $(4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$ represents family of hyperbolas having as asymptotes the lines $x + y = 0$, $2x + y + 1 = 0$. **(15)**



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Q.2 (b) Find the values of a, b and c such that the maximum value of directional derivative of $f = axy^2 + byz + cx^2z^2$ at $(1, -1, 1)$ is in the direction parallel to y axis and has magnitude 6. **(20)**



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Q.2 (c) Three forces P, Q and R act along the sides BC, CA and AB of ΔABC in order to keep the system in equilibrium. If the resultant force touches the inscribed circle, then prove that

$$\frac{1+\cos\alpha}{P}+\frac{(1+\cos\beta)}{Q}+\frac{1+\cos\gamma}{R}=0. \qquad (15)$$

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Q.3 (a) Find singular solution of the differential equation $y^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right)$

$\tan^2\beta + y^2\sec^2\beta - x^2\tan^2\beta = 0$ directly and from its complete primitive. Determine tac locus. Show that the envelope of family of curves, which is represented by the given equation, is $y = \pm x\tan\beta$. **(20)**



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Q.3 (b) A person is drawing water from a well with a light bucket which leaks uniformly. The bucket weighs 50 kg when it is full. When it arrives at the top, half of the water remains inside. If the depth of the water level in the well from the top is 30m, then find the work done in raising the bucket to the top form the water level. **(15)**



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Q.3 (c) Prove that the central orbit is a plane curve. (15)



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Q.4 (a) Given that $y_1 = x^2$ is a solution of the differential equation, $x^3\frac{d^2y}{dx^2} - (x^2+3x)\frac{dy}{dx} + 6y = 0$, find the other linearly independent solution of the above differential equation and write down the general solution of the differential equation. **(15)**

Let $y = v(x)$ be such that $y = u(x)\,v(x)$ is the general solution of the given ODE where $u(x) = x^2$ is one solution.

Converting given D.E. by changing the dependent variable y, we get

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\right)\frac{dv}{dx} = \frac{R}{u}$$

for given ODE,

$$\frac{d^2y}{dx^2} - \left(\frac{1}{x} + \frac{3}{x^2}\right)\frac{dy}{dx} + \frac{6}{x^3}y = 0,$$

$$P = -\left(\frac{1}{x} + \frac{3}{x^2}\right), \quad R = 0.$$

$$\therefore P + \frac{2}{u}\frac{du}{dx} = \frac{-1}{x} - \frac{3}{x^2} + \frac{2}{x^2}\cdot 2x$$

$$= 3\left(\frac{1}{x} - \frac{1}{x^2}\right)$$

$$\therefore \frac{d^2v}{dx^2} + 3\left(\frac{1}{x} - \frac{1}{x^2}\right)\frac{dv}{dx} = 0$$

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$$\Rightarrow \frac{dV}{dx} + \left(\frac{3}{x} - \frac{3}{x^2}\right) V = 0 \qquad \left\{ \text{Let } V = \frac{dv}{dx} \right\}$$

$$\Rightarrow \int \frac{dV}{V} = \int \left(\frac{3}{x^2} - \frac{3}{x}\right) dx$$

$$\Rightarrow \ln V = \frac{-3}{x} - 3\ln x + C_1$$

$$\Rightarrow V = \frac{C_2 e^{-3/x}}{x^3} \Rightarrow \int dv = \int \frac{C_2 e^{-3/x}}{x^3} dx$$

Put $\frac{-3}{x} = t \Rightarrow \frac{3}{x^2} dx = dt$

$$\therefore v = \int \frac{C_2 e^{-3/x}}{x^3} dx = \frac{-1}{9} \int t e^{-t} dt \cdot (C_2)$$

$$\Rightarrow v = \frac{-1}{9} \left\{ \frac{t e^{-t}}{(-1)} + \int e^{-t} dt \right\} (C_2)$$

$$\Rightarrow v = \frac{1}{9} \left\{ t e^{-t} + e^{-t} + C_3 \right\} (C_2)$$

$$\Rightarrow v = C_2' \left(\frac{-3}{x} e^{3/x} + e^{3/x} + C_3 \right)$$

$\therefore$ general solution is

$$y = uv = C_2' \left(-3x e^{3/x} + x^2 e^{3/x} + C_3 x^2 \right)$$

$$= A e^{3/x} x (x-3) + B x^2$$

where A & B are arbitrary constants.

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$\Rightarrow$ $\boxed{w(x) = x e^{3/x}(x-3)}$

is another linearly independent solution

And general solution is

$$\boxed{y = A x(x-3) e^{3/x} + B x^2}$$

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Q.4 (b) Let $F = \dfrac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$. (a) Calculate $\nabla \times F$. (b) Evaluate $\oint F.dr$ around any closed path and explain the results. **(20)**

(a) $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}$

$= \hat{i}(0) - \hat{j}(0) + \hat{k}\left(\dfrac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} + \dfrac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2}\right)$

$\Rightarrow \nabla \times \vec{F} = 0$

(b) $\oint \vec{F} \cdot d\vec{r} = \oint \dfrac{-y\,dx}{x^2+y^2} + \dfrac{x\,dy}{x^2+y^2}$

Converting to polar coordinates,

$x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow dx = \cos\theta\,(dr) - r\sin\theta\,d\theta$

And $dy = \sin\theta\,(dr) + r\cos\theta\,d\theta$

$\therefore \oint \vec{F} \cdot d\vec{r} = \oint \dfrac{(-r\sin\theta)(\cos\theta\,dr - r\sin\theta\,d\theta) + (r\cos\theta)(\sin\theta\,dr + r\cos\theta\,d\theta)}{r^2}$

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$\Rightarrow \vec{F}\cdot d\vec{r} = \oint d\theta$

$\therefore$ If the closed curve does not contain origin then $\boxed{\oint \vec{F}\cdot d\vec{r} = 0}$

If the closed curve contains the origin then $\boxed{\oint \vec{F}\cdot d\vec{r} = 2\pi}$

Explaination:

If $\nabla \times \vec{F}$ is 0 for a vector field $\vec{F}$ in a region R then $\vec{F} = \nabla\phi$ and $\oint \vec{F}\cdot d\vec{r} = 0$ holds if R is a simply connected domain.

for $\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{k}$,

$\vec{F}$ is not defined at $(0,0)$

$\therefore$ $\nabla \times \vec{F}$ is not entirely zero in any domain containing $(0,0)$.

In any other region not containing $(0,0)$, $\nabla \times \vec{F} = 0 \Rightarrow \oint \vec{F}\cdot d\vec{r} = 0$ over any such closed path.

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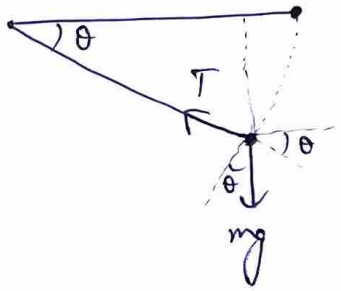
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Q.4 (c) A particle attached to a fixed peg O by a string of length l, is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with the horizontal, the resultant acceleration is $g\sqrt{1+3\sin^2\theta}$. (15)

Tangential acceleration

$$a_T = \frac{mg\cos\theta}{m} = g\cos\theta$$



Now, $T - mg\sin\theta = \frac{mv^2}{l}$

By conservation of Energy :-

$$0 + 0 = \frac{1}{2}mv^2 - mgl\sin\theta$$

$$\Rightarrow v^2 = 2gl\sin\theta$$

$$\Rightarrow a_{radial} = \frac{v^2}{l} = 2g\sin\theta$$

$$\therefore a_{total} = \sqrt{a_T^2 + a_{radial}^2}$$

$$= \sqrt{g^2\cos^2\theta + 4g^2\sin^2\theta}$$

$$= g\sqrt{1+3\sin^2\theta}$$

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SECTION-B

Q.5 (a) Solve the initial value problem $(2x^2 + y)dx + (x^2y - x)dy = 0$, $y(1) = 2$. **(10)**

$M = 2x^2 + y$

$N = x^2y - x$

$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 - 2xy + 1$

$= 2(1 - xy)$

$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2}{x} = f(x)$

$\therefore I.F = e^{\int \frac{-2}{x} dx} = \frac{1}{x^2}$

$\therefore$ D.E. becomes

$2\,dx + \frac{y}{x^2}dx + y\,dy - \frac{1}{x}dy = 0$

$\Rightarrow 2x + \frac{y^2}{2} - \frac{y}{x} = c$

$\Rightarrow 4x^2 + xy^2 - 2y = 2cx$

$y(1) = 2$

$\Rightarrow c = 2$

$\Rightarrow \boxed{4x^2 + xy^2 - 4x - 2y = 0}$

is the solution.

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Q.5 (b) If the unit vector $\bar{t}$ and binormal $\bar{b}$ makes angle θ and ϕ respectively with a constant unit vector $\bar{a}$, prove that $\frac{\sin\theta}{\sin\phi}\cdot\frac{d\theta}{d\phi}=-\frac{\kappa}{\tau}$. (10)

$\bar{T}\cdot\bar{a}=\cos\theta$

And $\bar{b}\cdot\bar{a}=\cos\phi$

$\Rightarrow \frac{d\bar{T}}{ds}\cdot\bar{a}=-\sin\theta\frac{d\theta}{ds}$ — (1) { s is arc length of curve }

And $\frac{d\bar{b}}{ds}\cdot\bar{a}=-\sin\phi\frac{d\phi}{ds}$ — (2)

Dividing (1) by (2),

$$\frac{\sin\theta\,d\theta}{\sin\phi\,d\phi}=\frac{\frac{d\bar{T}}{ds}\cdot\bar{a}}{\frac{d\bar{b}}{ds}\cdot\bar{a}}$$

$$=\frac{(K\bar{n})\cdot\bar{a}}{(-\tau\bar{n})\cdot\bar{a}} \quad \left\{\because \frac{d\bar{T}}{ds}=K\bar{n} \;\&\; \frac{d\bar{b}}{ds}=-\tau\bar{n}\right\}$$

where $\bar{n}$ is unit normal vector

$\Rightarrow \frac{\sin\theta}{\sin\phi}\cdot\frac{d\theta}{d\phi}=-\frac{K}{\tau}$.

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Q.5 (c) Prove that $\nabla \times (\phi A) = (\nabla \phi) \times A + \phi(\nabla \times A)$ **(10)**

$$\nabla \times (\phi \vec{A}) = \sum \hat{i} \times \frac{\partial}{\partial x}(\phi \vec{A}) \quad \left\{ \because \nabla \times \vec{B} = \sum \hat{i} \times \frac{\partial \vec{B}}{\partial x} \right\}$$

$$= \sum \hat{i} \times \left(\frac{\partial \phi}{\partial x} \vec{A} + \phi \frac{\partial \vec{A}}{\partial x} \right)$$

$$= \sum \hat{i} \times \left(\left(\frac{\partial \phi}{\partial x} \right) \vec{A} \right) + \sum \hat{i} \times \left(\phi \frac{\partial \vec{A}}{\partial x} \right)$$

$$= \sum \left(\frac{\partial \phi}{\partial x} \hat{i} \right) \times \vec{A} + \phi \sum \hat{i} \times \frac{\partial \vec{A}}{\partial x}$$

$$= \left(\sum \frac{\partial \phi}{\partial x} \hat{i} \right) \times \vec{A} + \phi (\nabla \times \vec{A})$$

$$= (\nabla \phi) \times \vec{A} + \phi (\nabla \times \vec{A})$$

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Q.5 (d) A particle is projected in a direction making an angle α with the horizontal. It passes through the two points (x_1, y_1) and (x_2, y_2). Prove that $\tan\alpha = \frac{y_1 R}{Rx_1 - x_1^2} = \frac{x_2^2 y_1 - x_1^2 y_2}{(x_1 x_2)(x_2 - x_1)}$, where R denotes the horizontal range. **(10)**

$$\frac{u^2 \sin 2\alpha}{g} = R \Rightarrow \frac{u^2}{g} = R \operatorname{cosec} 2\alpha \quad \text{— (1)}$$

Now, equation of projectile's path is

$$y = x\tan\alpha - \frac{g x^2}{2u^2 \cos^2\alpha}$$

$$\Rightarrow y = x\tan\alpha - \frac{x^2 (2\sin\alpha\cos\alpha)}{2R\cos^2\alpha} \quad \{\text{from (1)}\}$$

$$\Rightarrow y = x\tan\alpha - \frac{x^2}{R}\tan\alpha$$

$$\Rightarrow y = \tan\alpha\left(\frac{Rx - x^2}{R}\right)$$

$$\Rightarrow \tan\alpha = \frac{yR}{Rx - x^2}$$

$\because (x_1, y_1)$ lies on the path

$$\Rightarrow \tan\alpha = \frac{y_1 R}{Rx_1 - x_1^2} \quad \text{— (1)}$$

Similarly, $\tan\alpha = \frac{y_2 R}{Rx_2 - x_2^2}$

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$$\Rightarrow \frac{y_1}{Rx_1 - x_1^2} = \frac{y_2}{Rx_2 - x_2^2}$$

$$\Rightarrow R(x_1y_2 - x_2y_1) = x_1^2y_2 - x_2^2y_1 \quad \text{— (2)}$$

$$\therefore \tan\alpha = \frac{y_1\left(x_1^2y_2 - x_2^2y_1\right)}{x_1y_2 - x_2y_1} \Big/ x_1\left(x_1^2y_2 - x_2^2y_1\right) - x_1^2$$

$$= \frac{y_1\left(x_1^2y_2 - x_2^2y_1\right)}{x_1^3y_2 - x_1x_2^2y_1 - x_1^3y_2 + x_1^2x_2y_1}$$

$$\Rightarrow \tan\alpha = \frac{x_2^2y_1 - x_1^2y_2}{x_1x_2(x_2 - x_1)} \quad \text{— (3)}$$

From (1) & (3), we have the required result.

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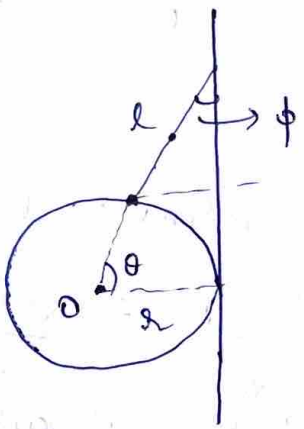
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Q.5 (e) A cylinder of radius 'r' whose axis is fixed horizontally, touches a vertical wall along a generating line. A flat beam of length 'l' and weight W rests with its extremities in contact with the wall and the cylinder, making an angle 45° with the vertical. Prove that the reaction of the cylinder is $\frac{W\sqrt{5}}{2}$ and the pressure on the wall is $W/2$. Also prove that the ratio of radius of the cylinder to the length of the beam is $(5+\sqrt{5}):4\sqrt{2}$. **(10)**

We have

~~$l\cos\phi +$~~

$l\sin\phi + r\cos\theta = r$ — (1)



Also, height of Centre of Mass of beam from O

$z = r\sin\theta + \frac{l}{2}\cos\phi$

The only force doing work is gravity

∴ By principle of virtual work

$\delta z = 0 \Rightarrow r\cos\theta\,\delta\theta - \frac{l}{2}\sin\phi\,\delta\phi = 0$

From (1), $l\cos\phi\,\delta\phi - r\sin\theta\,\delta\theta = 0$

$\Rightarrow r\cos\theta\,\delta\theta - \frac{l}{2}\sin\phi\left(\frac{r\sin\theta}{l\cos\phi}\right)\delta\theta = 0$

$\Rightarrow \cos\theta = \frac{\tan\phi\sin\theta}{2}$

Also, $\phi = 45° \Rightarrow \tan\theta = 2$

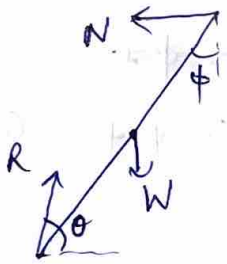
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$\therefore$ From (1), $\frac{l}{\sqrt{2}} + \frac{\lambda}{\sqrt{5}} = \lambda$

$\Rightarrow \frac{l}{\sqrt{2}} = \lambda\left(\frac{\sqrt{5}-1}{\sqrt{5}}\right)$

$\Rightarrow \frac{\lambda}{l} = \frac{\sqrt{5}}{\sqrt{2}(\sqrt{5}-1)} = \frac{5+\sqrt{5}}{4\sqrt{2}}$



$\therefore R\sin\theta = W$

$\Rightarrow R = \frac{\sqrt{5}W}{2}$ (reaction of cylinder) $\left\{\because \sin\theta = \frac{2}{\sqrt{5}}\right\}$

And $N = R\cos\theta$

$\Rightarrow N = \frac{\sqrt{5}W}{2} \cdot \frac{1}{\sqrt{5}} = \frac{W}{2}$

(pressure on the wall)

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Q.6 (a) Reduce $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$, where P, Q, R are functions of x, to normal form. Hence solve $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)y=-3e^{x^2}\sin 2x$. **(20)**

Let $y = uv$ be the complete solution of $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ — (1)

$$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \quad — (2)$$

And, $$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\left(\frac{du}{dx}\right)\left(\frac{dv}{dx}\right) + v\frac{d^2u}{dx^2}$$

$\therefore$ (1) becomes

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx}+Pu\right)\frac{dv}{dx} + \left(P\frac{du}{dx}+\frac{d^2u}{dx^2}+Qu\right)v = R$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} + \left(Q+\frac{P}{u}\frac{du}{dx}+\frac{1}{u}\frac{d^2u}{dx^2}\right)v = \frac{R}{u} \quad — (3)$$

for, normal form,

$$P+\frac{2}{u}\frac{du}{dx}=0$$

$$\Rightarrow \frac{du}{u} = \frac{-P\,dx}{2}$$

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$$\Rightarrow u = e^{-\int \frac{P}{2} dx}$$

Also, $Q + \frac{P}{u}\frac{du}{dx} + \frac{1}{u}\frac{d^2u}{dx^2}$

$$= Q + \frac{P}{u}\left(-\frac{Pu}{2}\right) + \frac{1}{u}\left(-\frac{1}{2}\left(P\frac{du}{dx} + u\frac{dP}{dx}\right)\right)$$

$$= Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$$

$\therefore$ Normal form becomes

$$\frac{d^2v}{dx^2} + Iv = s$$

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

and $s = \frac{R}{u}$; where $u = e^{-\frac{1}{2}\int P dx}$

For $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x$

$$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$$

$$= 4x^2 - 1 - \frac{1}{4}\cdot 16x^2 - \frac{1}{2}(-4)$$

$$= 1$$

And, $s = \frac{R}{u} = \frac{-3e^{x^2}\sin 2x}{e^{-\frac{1}{2}\int(-4x)dx}} = -3\sin 2x$

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$\therefore \quad \frac{d^2v}{dx^2} + v = -3\sin 2x$

This is a Linear ODE in constant coefficients

Auxillary Equation is

$m^2+1 = 0 \Rightarrow m = \pm i$

$\therefore$ Complementary function $v_c = A\cos x + B\sin x$

And P.I. $= \frac{1}{(D^2+1)}(-3\sin 2x)$

$= \frac{-3}{(D^2+1)}(\sin 2x)$

$= \sin 2x$

$\therefore \quad v(x) = A\cos x + B\sin x + \sin 2x$

where A, B are arbitrary constants.

$\therefore$ General solution of given D.E. is

$$\boxed{y = uv = e^{x^2}(A\cos x + B\sin x + \sin 2x)}$$

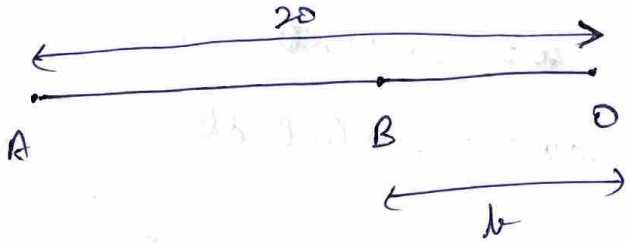
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Q.6 (b) A particle of mass 5 units moves in a straight line towards a centre of force and the force varies inversely as the cube of distance. Starting from rest at the point A distant 20 units from centre of force O, it reaches a point B distant b from O. Find the time in reaching from A to B and the velocity at B. When will the particle reach at the centre? **(15)**



Acceleration $a = \frac{-K}{x^3}$ {K is constant}

{Let line be x axis, O the origin and ABO be the -ve x direction}

$\Rightarrow \frac{d^2x}{dt^2} = \frac{-K}{x^3}$

$\Rightarrow \int v\,dv = \int \frac{-K}{x^3}\,dx$ $\left\{\because v\,dv = a\,dx\right\}$

$\Rightarrow \frac{v^2}{2} = \frac{K}{2x^2} + A$

For $x = 20$, $v = 0$

$\Rightarrow A = \frac{-K}{800}$

$\therefore v^2 = \frac{K}{x^2} - \frac{K}{400}$ ——— (1)

$\Rightarrow \frac{dx}{dt} = -\sqrt{\frac{K}{x^2} - \frac{K}{400}}$

$\Rightarrow \int \frac{dx}{\sqrt{\frac{K}{x^2} - \frac{K}{400}}} = \int -dt$

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$$\Rightarrow \frac{1}{\sqrt{K}} \int \frac{20x\,dx}{\sqrt{400-x^2}} = -t + B$$

$\Rightarrow$ Put $x = 20\sin\theta$

$\Rightarrow dx = 20\cos\theta\,d\theta$

$$\Rightarrow \frac{1}{\sqrt{K}} \int \frac{20 \cdot 20\sin\theta \cdot 20\cos\theta\,d\theta}{20\cos\theta} = -t + B$$

$$\Rightarrow -\frac{400\cos\theta}{\sqrt{K}} = -t + B$$

$$\Rightarrow \frac{400}{\sqrt{K}}\sqrt{1-\frac{x^2}{400}} = t - B$$

$$\Rightarrow \frac{20}{\sqrt{K}}\sqrt{400-x^2} = t - B$$

for $x = 20$, $t = 0$

$\Rightarrow B = 0$

$$\therefore t = \frac{20}{\sqrt{K}}\sqrt{400-x^2}$$

Time in reaching A to B :

$$t_1 = \frac{20}{\sqrt{K}}\sqrt{400-b^2}$$

Velocity at B, $v_B^2 = \frac{K}{x^2} - \frac{K}{400}$

$$\Rightarrow v_B = \sqrt{\frac{K}{b^2} - \frac{K}{400}}$$

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$$\Rightarrow \boxed{v_B = \frac{\sqrt{(400 - b^2)k}}{20b}}$$

Time in reaching the centre (O) :

$$T = \frac{20}{\sqrt{k}} \sqrt{400 - x^2} = \boxed{\frac{400}{\sqrt{k}} \text{ units}}$$



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Q.6 (c) If α and β are the two possible directions to hit a given point (a, b), then show that $\tan(\alpha+\beta)=-\frac{a}{b}$. **(15)**

Equation of projectile's path is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

for (a, b) to be hit,

$$b = a\tan\theta - \frac{ga^2(1+\tan^2\theta)}{2u^2}$$

$$\Rightarrow \frac{ga^2}{2u^2}\tan^2\theta - a\tan\theta + \left(\frac{ga^2}{2u^2}+b\right)=0$$

$\tan\alpha$ and $\tan\beta$ are two solutions of the above quadratic

$$\therefore \tan\alpha + \tan\beta = \frac{a}{\frac{ga^2}{2u^2}} = \frac{2u^2}{ag}$$

And, $\tan\alpha\tan\beta = 1 + \frac{2u^2b}{ga^2}$

$$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{\frac{2u^2}{ag}}{1-\left(1+\frac{2u^2b}{ga^2}\right)}$$

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$$\Rightarrow \boxed{\tan(\alpha+\beta) = \frac{-a}{b}}$$



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Q.7 (a) Solve the differential equation: $(1-x^2)\frac{d^2y}{dx^2}-4x\frac{dy}{dx}-(1+x^2)y=x.$ (15)



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Q.7 (b) Find the radii of curvature and torsion at a point of intersection of the surfaces $x^2 - y^2 = c^2, y = x\tan h\left(\frac{z}{c}\right)$. **(20)**



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Q.7 (c) Find the law of force for the orbit $r^2 = a^2 \cos 2\theta$ (the pole being the centre of the force). (15)

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Q.8 (a) Solve the following initial value problem by using the Laplace transform technique:

$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y(t) = f(t)$, $y(0) = 1, y'(0) = 0$ and $f(t)$ is a given function of t. **(15)**

Taking Laplace Transform both sides,

$$p^2Y(p) - p\,y(0) - y'(0) - 4\left(pY(p) - y(0)\right) + 3Y(p) = F(p)$$

$\left\{\text{Let } L\{y(t)\} = Y(p) \text{ & } L\{f(t)\} = F(p)\right\}$

$$\Rightarrow (p^2 - 4p + 3)Y(p) - p + 4 = F(p)$$

$$\Rightarrow Y(p) = \frac{F(p) + p - 4}{(p^2 - 4p + 3)}$$

$$\Rightarrow y(t) = L^{-1}\{Y(p)\}$$

$$= L^{-1}\left\{F(p)\cdot\frac{1}{p^2-4p+3} + \frac{p-4}{(p^2-4p+3)}\right\}$$

$$= L^{-1}\left\{F(p)\cdot\frac{1}{(p-2)^2-1^2} + \frac{(p-2)}{(p-2)^2-1^2} - \frac{2}{(p-2)^2-1^2}\right\}$$

$$= f(t) * \left(e^{2t}\sinh t\right) + e^{2t}\cosh t - 2e^{2t}\sinh t$$

where * is convolution operator

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$$e^{2t}\sinh t = \frac{e^{3t} - e^{t}}{2}$$

$$e^{2t}\cosh t = \frac{e^{3t} + e^{t}}{2}$$

$$\therefore y(t) = f(t) * \left(\frac{e^{3t} - e^{t}}{2}\right) + \frac{e^{3t} + e^{t}}{2}$$

$$\Rightarrow y(t) = \int_0^t \left(\frac{e^{3u} - e^{u}}{2}\right) f(t-u)\,du + \frac{3}{2}e^{t} - \frac{e^{3t}}{2}$$

$$+e^{t} - e^{3t}$$

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Q.8 (b) Given that C is a curve of intersection of the cylinder $x^2+y^2=4$ and the plane $x+y+z=2$ and C is described counterclockwise. Verify Stokes Theorem for the line integral $\int_C(-y^3dx+x^3dy-z^3dz)$. **(15)**

Let $\vec{F} = -y^3\hat{i} + x^3\hat{j} - z^3\hat{k}$

$$\Rightarrow \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix}$$

$$= 3(x^2+y^2)\hat{k}$$

Also, if we consider the surface of the plane $x+y+z=2$ which has $x^2+y^2=4$ as its boundary, (say S),

$$\hat{n} = \nabla(x+y+z-2) \Big/ |\nabla(x+y+z-2)|$$

$$= \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\therefore (\nabla\times\vec{F})\cdot\hat{n} = \sqrt{3}(x^2+y^2)$$

$$\therefore \iint_S \sqrt{3}(x^2+y^2)\,ds$$

$$= \iint_R \sqrt{3}(x^2+y^2)\frac{dxdy}{|\hat{n}\cdot\hat{k}|}$$

{ R is projection of S on x-y plane }

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R is the circular disc $x^2+y^2 \le 4$ in the x-y plane.

$$\therefore \iint_S \sqrt{3}\,(x^2+y^2)\,ds = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 3r^2 \cdot r\,dr\,d\theta$$

$$= \frac{6\pi}{4}\cdot 2^4$$

$$= \boxed{24\pi}$$

Line Integral

NEXT IAS

Let $x = 2\cos\theta$ $y = 2\sin\theta$

and $z = 2 - x - y$

$= 2(1 - \cos\theta - \sin\theta)$

$\Rightarrow dx = -2\sin\theta\,d\theta$

$dy = 2\cos\theta\,d\theta$

$dz = 2\sin\theta\,d\theta - 2\cos\theta\,d\theta$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_{\theta=0}^{2\pi} -(8\sin^3\theta)(-2\sin\theta\,d\theta) + (8\cos^3\theta)(2\cos\theta\,d\theta) - 8(1-\cos\theta-\sin\theta)^3 (2\sin\theta - 2\cos\theta)\,d\theta$$

$$= 16\int_0^{2\pi}(\sin^4\theta + \cos^4\theta)\,d\theta$$

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$$= 16 \times 8 \int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$= 16 \times 8 \times \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \boxed{24\pi}$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS$$

$\therefore$ Stokes' theorem is verified.

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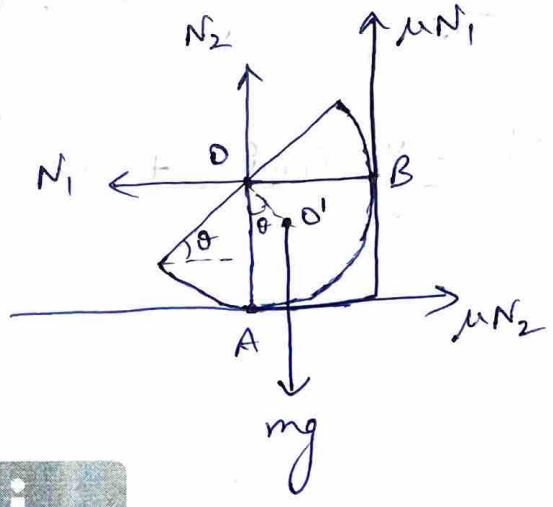
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Q.8 (c) A semi-circular disc rests in a vertical plane with its curved edge on a rough horizontal and equally rough vertical plane, the coefficient of friction being μ. Show that the greatest angle that the bounding diameter can make with the horizontal plane is $\sin^{-1}\left(\frac{3\pi}{4}\frac{\mu+\mu^2}{1+\mu^2}\right)$.

(20)

When the disc is about to slip, both frictional forces will be maximum possible



Centre of mass of disc is at $OO' = \frac{4R}{3\pi}$

Putting Moment about A $= 0$ (R is radius of disc)

$$(\mu N_1)R + N_1 R - mg\left(\frac{4R}{3\pi}\right)\sin\theta = 0$$

$$\Rightarrow N_1(1+\mu) = \frac{4mg}{3\pi}\sin\theta \quad \text{— (1)}$$

Moment about B $= 0$

$$\Rightarrow (\mu N_2)R - N_2 R + mg\left(R - \frac{4R}{3\pi}\sin\theta\right) = 0$$

$$\Rightarrow N_2(1-\mu) = mg\left(1 - \frac{4\sin\theta}{3\pi}\right) \quad \text{— (2)}$$

Now, by principle of virtual work,

(work here is done by gravity & the two frictional forces with change in θ)

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$$-\delta\left(\left(R-\frac{4R}{3\pi}\cos\theta\right)mg\right)+\delta\left(\mu N_1 R\theta+\mu N_2 R\theta\right)=0$$

$$\delta\theta\left(-\frac{4}{3\pi}\sin\theta\,(mg)+\mu(N_1+N_2)\right)=0$$

$$\Rightarrow \frac{-4}{3\pi}\sin\theta+\mu\left(\frac{4\sin\theta}{3\pi(1+\mu)}+\frac{1}{(1-\mu)}\left(1-\frac{4\sin\theta}{3\pi}\right)\right)=0$$

$$\Rightarrow -\sin\theta+\frac{\mu\sin\theta}{1+\mu}+\frac{3\pi}{4}\frac{\mu}{1-\mu}-\frac{\mu\sin\theta}{1-\mu}=0 \quad \{\text{from } ① \& ②\}$$

$$\Rightarrow \sin\theta\left(1-\frac{\mu}{1+\mu}+\frac{\mu}{1-\mu}\right)=\frac{3\pi}{4}\frac{\mu}{(1-\mu)}$$

$$\Rightarrow \sin\theta\left(\frac{1-\mu^2-\mu(1-\mu)+\mu(1+\mu)}{(1-\mu^2)}\right)=\frac{3\pi}{4}\frac{\mu}{1-\mu}$$

$$\Rightarrow \sin\theta\,\frac{(1+\mu^2)}{(1+\mu)}=\frac{3\pi\mu}{4}$$

$$\Rightarrow \sin\theta=\frac{3\pi}{4}\frac{\mu+\mu^2}{1+\mu^2}$$

$$\Rightarrow \boxed{\theta=\sin^{-1}\left(\frac{3\pi}{4}\cdot\frac{\mu+\mu^2}{1+\mu^2}\right)}$$

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