

NEXT IAS

(To be filled by candidate)

Name of Candidate : MADHUKANT

Roll No. : Date of Examination : 18/07/24

Mobile Number : E-mail :

Exam Centre : Old Rajinder Nagar ☐

Online ☒

CSE (MAINS) TEST SERIES-2024

Test-03	MATHEMATICS
Dated : 07-07-2024	OPTIONAL

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are **EIGHT** questions divided in **Two** sections and printed in **ENGLISH**.

Candidate has to attempt **FIVE** questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

DELHI CENTRE:

Old Rajinder Nagar: 27-B, Pusa Road, Metro Pillar no. 118, Near Karol Bagh Metro, New Delhi-110060 | Phone: 8081300200

Mukherjee Nagar: 1422, Main Mukherjee Nagar Road, Near Batra Cinema, New Delhi-110009 | Phone: 8081300200

BHOPAL CENTRE: Plot No. 46 Zone - 2, M.P Nagar, Bhopal - 462011 | Phone: 8827664612, 8081300200

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	20			
2. (c)	21	15			
3. (a)	24	20			
3. (b)	28	15			
3. (c)	31	15			
4. (a)	34	15			
4. (b)	37	20			
4. (c)	41	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	20			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	81	20			
Grand Total					

Remarks

Observations:

SECTION-A

Q.1 (a) Find the singular solution of the differential equation $9p^2(2-y)^2 = 4(3-y)$.

(10)

Given diff. eqⁿ: $9p^2(2-y)^2 = 4(3-y)$





Q.1 (b)

Using Laplace Transform, solve the IVP $ty'' + 2ty' + 2y = 2$; $y(0) = 1$ and $y'(0)$ is arbitrary. Does this problem has a unique solution. (10)

Given: $ty'' + 2ty' + 2y = 2$ ① $y(0) = 1$
 $y'(0) = a$ (arbitrary)

Let $L\{y\} = f(p)$

Wkt, $L\{y'\} = pf(p) - y(0) = pf(p) - 1$

$L\{ty'\} = -\frac{d}{dp}(pf(p) - 1) = -f(p) - pf'(p)$

And, $L\{y''\} = p^2 f(p) - py(0) - y'(0)$

$= p^2 f(p) - p - a$

$L\{ty''\} = -\frac{d}{dp}(p^2 f(p) - p - a)$

$= -2pf(p) - p^2 f'(p) + 1$

Now, Applying Laplace in eqn ① $\rightarrow$

$L\{ty''\} + 2L\{ty'\} + 2L\{y\} = L\{2\}$

$-2pf(p) - p^2 f'(p) + 1 + 2(-f(p) - pf'(p))$

$+ 2f(p) = \frac{2}{p}$

$-f'(p)(p^2 + 2p) - 2pf(p) = \frac{2}{p}$

$f'(p) + \frac{2p}{p^2 + 2p} f(p) = -\frac{2}{p(p^2 + 2p)}$ ②

thus is linear diff. eqn

$$\therefore \text{Integrating factor} = e^{\int \frac{2}{p+2} dp} = e^{2 \log(p+2)} \\ = (p+2)^2$$

$\therefore$ Solⁿ for diff. eqⁿ in (2)

$$f(p) \times (p+2)^2 = \int \frac{-2}{p(p+2)} \times (p+2)^2 dp \\ = \int -\frac{2(p+2)}{p^2} dp \\ = \int -\frac{2}{p} - \frac{4}{p^2} dp = -2 \log p + \frac{4}{p} + C$$

$$\Rightarrow f(p) = -2 \frac{\log p}{(p+2)^2} + \frac{4}{p(p+2)^2} + \frac{C}{(p+2)^2}$$

$$y(t) = L^{-1}\{f(p)\} =$$



Q.1 (c)

Evaluate $\int_{0,0}^{2,1} (10x^4 - 2xy^3)dx - 3x^2y^2dy$ along the path $x^4 - 6xy^3 = 4y^2$.

(10)

To evaluate: $I = \int_{0,0}^{2,1} (10x^4 - 2xy^3)dx - 3x^2y^2dy$

along path $x^4 - 6xy^3 = 4y^2$

let $M = 10x^4 - 2xy^3$ & $N = -3x^2y^2$

then $I = \int_{0,0}^{2,1} Mdx + Ndy$

& $\frac{\partial M}{\partial y} = -6xy^2$ & $\frac{\partial N}{\partial x} = -6xy^2 = \frac{\partial M}{\partial y}$

$\therefore Mdx + Ndy$ is exact

$\therefore Mdx + Ndy$ is independent of path

& $\exists \phi$ s.t. $d\phi = Mdx + Ndy$

$\phi = \int Mdx + \int Ndy$
only y-terms.

$= \int (10x^4 - 2xy^3)dx + \int 0dy$

$= 2x^5 - x^2y^3$

$\therefore I = \int_{0,0}^{2,1} d(2x^5 - x^2y^3)$

$= 2(2)^5 - 2^2(1)^3 = 64 - 4 = \boxed{60}$

Ans



Q.1 (d)

If mass density at any point of a cord varies as the radius of curvature of the curve in which it hangs freely under gravity, then prove that this curve is the catenary of uniform strength.

(10)



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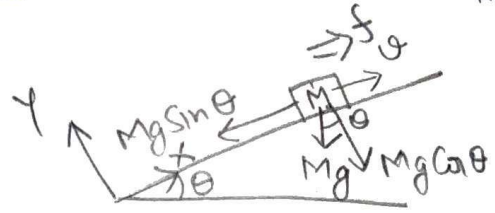


Q.1 (e)

A train of mass M kg is ascending a smooth incline of 1 in n and velocity is v m/s, its acceleration is f m/s². Prove that the effective power of the engine is $Mv(nf + g)/n$ watts. (10)

let Mass M ascends on plane
with angle of Inclination θ

At any point, Assume
its velocity = v ms⁻¹
acceleration = f ms⁻²



$$\text{incline is } 1 \text{ in } n \Rightarrow \sin \theta = \frac{1}{n} \quad \text{--- (1)}$$

let F be force generated by engine

$$\text{then net acceleration } \Rightarrow Mf = F - Mg \sin \theta$$

$$\Rightarrow F = Mf + Mg \sin \theta$$

small work done in moving train δx will be $\rightarrow$

$$dW = F \delta x = M(f + g \sin \theta) \delta x$$

$$\& \text{ Power generated} = \frac{dW}{dt} = \frac{F \delta x}{\delta t}$$

$$= M(f + g \sin \theta) \frac{\delta x}{\delta t}$$

$$= M(f + g \sin \theta) v \quad \left(\text{as } \delta x, \delta t \rightarrow 0 \right)$$

$$= M\left(f + \frac{g}{n}\right) v \quad \left(\text{from eq (1)} \right)$$

$$\Rightarrow \boxed{\text{Power} = M(f + g/n) v}$$

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Q.4 (a)

Given that $y_1 = x^2$ is a solution of the differential equation, $x^3 \frac{d^2 y}{dx^2} - (x^2 + 3x) \frac{dy}{dx} + 6y = 0$, find the other linearly independent solution of the above differential equation and write down the general solution of the differential equation. (15)

Given differential eqⁿ:

$$x^3 \frac{d^2 y}{dx^2} - (x^2 + 3x) \frac{dy}{dx} + 6y = 0$$

$$\text{or } \frac{d^2 y}{dx^2} - \frac{(x+3)}{x^2} \frac{dy}{dx} + \frac{6}{x^3} y = 0 \quad \text{--- (1)}$$

On comparing with $\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Q y = 0$

$$P = -\frac{(x+3)}{x^2} \quad \& \quad Q = \frac{6}{x^3}$$

Consider $y_1 = x^2$. putting it in eq (1) $\rightarrow$

$$2 - \left(\frac{x+3}{x} \right) 2 + \frac{6}{x} = 0$$

$\therefore y_1 = x^2$ is solⁿ of eq (1)

Now, we know that, if $y = y_1 v$ is general solution of eqⁿ (1)

then a diff. eq of v is given by

$$\frac{d^2 v}{dx^2} + \left(P + \frac{2y_1'}{y_1} \right) \frac{dv}{dx} = 0$$

$$P + \frac{2y_1'}{y_1} = \frac{-(x+3)}{x^2} + \frac{2(2x)}{x^2}$$

$$= -\frac{1}{x} + \frac{3}{x^2} + \frac{4}{x} = 3 \left(\frac{1}{x} - \frac{1}{x^2} \right) = 3 \left(\frac{x-1}{x^2} \right)$$

∴ differential eqⁿ of $v \rightarrow$

$$\frac{dv}{dx^2} + 3\frac{(x-1)}{x^2} \frac{dv}{dx} = 0.$$

$$\frac{d(dv/dx)}{dv/dx} = -3\frac{(x-1)}{x^2} dx$$

$$\log\left(\frac{dv}{dx}\right) = \int -\frac{3}{x} + \frac{3}{x^2} dx + C_1 \quad (\text{on integrating})$$

$$\begin{aligned} \log\left(\frac{dv}{dx}\right) &= -3\log x + \frac{3(-1)}{x} + C_1 \\ &= \log \frac{1}{x^3} - \frac{3}{x} + C_1 \end{aligned}$$

$$\frac{dv}{dx} = e^{\log \frac{1}{x^3}} \cdot e^{-3/x} \cdot e^{C_1}$$

$$\frac{dv}{dx} = C_1 \frac{e^{-3/x}}{x^3}$$

$$v = C_1 \int \frac{e^{-3/x}}{x^3} dx + C_2$$

$$\text{let } \frac{1}{x} = t$$

$$= C_1 \int \frac{e^{-3t}(-x^2)dt}{(x^3)} + C_2 \quad \frac{1}{x^2} dx = dt$$

$$= C_1 \int \frac{e^{-3t}(-1)(t)dt}{1} + C_2$$

$$= -C_1 \int t e^{-3t} dt + C_2$$

$$= -C_1 \left[t \cdot \frac{e^{-3t}}{-3} - \frac{e^{-3t}}{9} \right] + C_2$$

$$= C_1 \left[\frac{t e^{-3t}}{3} + \frac{e^{-3t}}{9} \right] + C_2$$

$$= C_1 \left[\frac{e^{-3/x}}{3x} + \frac{e^{-3/x}}{9} \right] + C_2$$

∴ Solⁿ for diff. eqⁿ in eq (1) will be

$$y = y_1 \times y_2$$

$$y = x^2 \left[C_1 \left[\frac{e^{-3/x}}{3x} + \frac{e^{-3/x}}{9} \right] + C_2 \right]$$

$$= C_1 \left[\frac{x e^{-3/x}}{3} + \frac{x^2 e^{-3/x}}{9} \right] + C_2 x^2$$

$$= C_1 e^{-3/x} \left(\frac{x}{3} + \frac{x^2}{9} \right) + C_2 x^2$$

$$y = C_1 \frac{x e^{-3/x}}{3} \left(1 + \frac{x}{3} \right) + C_2 x^2$$

general solution

Q.4 (b)

Let $F = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$. (a) Calculate $\nabla \times F$. (b) Evaluate $\oint F \cdot d\mathbf{r}$ around any closed path and explain the results. (20)

Given: $\bar{F} = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$

(a) $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}$

$$= \hat{k} \left(\frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{y(2y)}{(x^2+y^2)^2} \right)$$

$$= \hat{k} \left(\frac{2}{x^2+y^2} - \frac{2}{x^2+y^2} \right) = 0$$

$$\therefore \boxed{\nabla \times \bar{F} = 0}$$

(b) to evaluate $\oint F \cdot d\mathbf{r}$ around any closed path

let C be any closed path as shown and S be surface enclosed by C in the same plane



We know that By Stokes theorem in C -

$$\oint_C F \cdot d\mathbf{r} = \iint_S (\nabla \times F) \cdot \hat{n} \, dS$$

But, we know that $\nabla \times \mathbf{F} = 0$

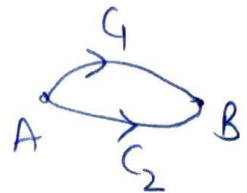
$$\Rightarrow \oint \mathbf{F} \cdot d\mathbf{r} = \iint_S 0 \, ds = 0$$

$\Rightarrow$ In any closed path $\oint \mathbf{F} \cdot d\mathbf{r} = 0$

Explanation

Now let us assume two points in space A & B

And C_1, C_2 are two different paths from A to B.



Now if we calculate $\oint \mathbf{F} \cdot d\mathbf{r}$ in closed loop of $A \rightarrow B$ via C_1 & $B \rightarrow A$ via C_2 i.e. $\oint_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r}$

then $\oint_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = 0$ (as shown above)

$$\Rightarrow \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} - \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

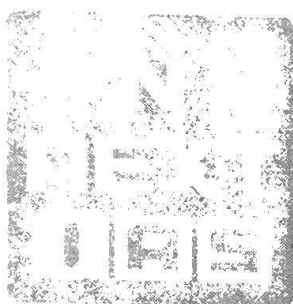
$$\Rightarrow \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$\Rightarrow$ Line Integral $\oint \mathbf{F} \cdot d\mathbf{r}$ is

independent of path chosen from A to B.

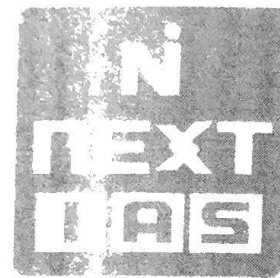
Such a field $\mathbf{F}$ is called Conservative field.

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Q.4 (c)

A particle attached to a fixed peg O by a string of length l , is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with the horizontal, the resultant acceleration is $g\sqrt{1+3\sin^2\theta}$. (15)

Let particle of mass m is attached by string of length l fixed to peg O .

Let m is lifted to A which is horizontal to peg O & let go

At A , velocity of particle $u_0 = 0$ and B be lowest point.

Let At Any point P , m makes θ angle with horizontal OA and tension in string is T

then force diagram will be as shown in fig. 1

tangential acceleration

$$m \frac{d^2 s}{dt^2} = -mg \cos \theta \quad \text{where } s \text{ is measured from } B, s \text{ is arc length.} \quad \text{--- (I)}$$

radial acceleration

$$\frac{mv^2}{l} = T - mg \sin \theta \quad \text{--- (II)} \quad v \text{ is velocity of } m.$$

we know that $s = l(\frac{\pi}{2} - \theta)$ $\frac{ds}{dt} = -l \frac{d\theta}{dt}$

$$\Rightarrow \frac{d^2 s}{dt^2} = -l \frac{d^2 \theta}{dt^2}$$

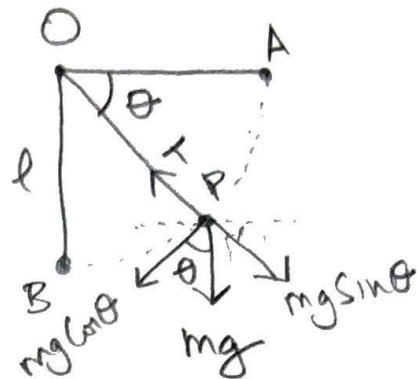


fig. 1

$\therefore$ In eq (1) $\rightarrow$

$$m \frac{d^2(\theta)}{dt^2} = -mg \sin \theta$$

$$-l \frac{d^2\theta}{dt^2} = -g \cos \theta \quad (\text{multiplying } d\theta/dt)$$

$$-(d\theta/dt) \frac{d(d\theta/dt)}{dt} = -\frac{g}{l} \cos \theta \frac{d\theta}{dt}$$

On integrating $-\frac{1}{2} \left(\frac{d\theta}{dt} \right)^2 = -\frac{g}{l} \sin \theta + B$

at $\theta = 0$ & $d\theta/dt = v_A = 0$

$\Rightarrow B = 0$

$$\Rightarrow -\frac{1}{2} \left(\frac{d\theta}{dt} \right)^2 = -\frac{g \sin \theta}{l}$$

$\therefore$ we know, $l \frac{d\theta}{dt} = v$

$\Rightarrow v^2 = +2gl \sin \theta$

$\therefore$ radial acceleration $= \frac{v^2}{l} = 2g \sin \theta$

tangential acceleration $= \frac{dv}{dt} = -g \cos \theta$

$\therefore$ resultant acceleration $= \sqrt{g^2 \cos^2 \theta + 4g^2 \sin^2 \theta}$
 $= g \sqrt{1 + 3 \sin^2 \theta}$

$\therefore$ Resultant accⁿ $= g \sqrt{1 + 3 \sin^2 \theta}$

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SECTION-B

Q.5 (a) Solve the initial value problem $(2x^2 + y)dx + (x^2y - x)dy = 0$, $y(1) = 2$.

Given: $(2x^2 + y)dx + (x^2y - x)dy = 0$
where $y(1) = 2$

Consider, $2x^2 dx + y dx + x^2y dy - x dy = 0$

dividing by $x^2 \rightarrow$

$$2dx + \frac{y dx - x dy}{x^2} + y dy = 0$$

$$2dx - d\left(\frac{y}{x}\right) + y dy = 0$$

On Integrating $\rightarrow$

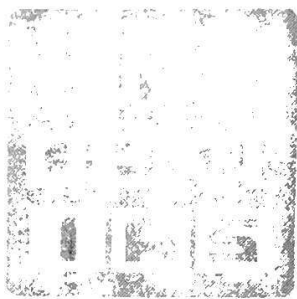
$$2x - \frac{y}{x} + \frac{y^2}{2} = C$$

at $x=1$, $y=2$

$$\Rightarrow 2 - 2 + \frac{4}{2} = C \rightarrow C = 2$$

$$\Rightarrow \boxed{2x - \frac{y}{x} + \frac{y^2}{2} = 2}$$

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Q.5 (b)

If the unit vector $\bar{t}$ and binormal $\bar{b}$ makes angle θ and ϕ respectively with a constant unit vector $\bar{a}$, prove that $\frac{\sin \theta}{\sin \phi} \cdot \frac{d\theta}{d\phi} = -\frac{\kappa}{\tau}$.

(10)

Given that: $\bar{t}$ is unit vector

$\bar{b}$ is binormal & hence unit vector

$\bar{a}$ is constant vector (unit)

$\therefore \bar{t}, \bar{b}$ make θ & ϕ respectively with $\bar{a}$

$$\Rightarrow \bar{t} \cdot \bar{a} = \cos \theta \quad \& \quad \bar{b} \cdot \bar{a} = \cos \phi \quad \text{--- (I)}$$

We know that, By Serret-Frenet equation

$$\frac{d\bar{t}}{ds} = \kappa \bar{N} \quad \text{and} \quad \frac{d\bar{b}}{ds} = -\tau \bar{N} \quad \text{--- (II)}$$

Now, differentiate eq (I) and (II) w.r.t s .

$$\frac{d\bar{t}}{ds} \cdot \bar{a} + \bar{t} \cdot \frac{d\bar{a}}{ds} = -\sin \theta \frac{d\theta}{ds} \quad \text{--- (III)}$$

$$\Rightarrow \frac{d\bar{t}}{ds} \cdot \bar{a} = -\sin \theta \frac{d\theta}{ds} \quad (\bar{a} \text{ is constant}) \quad \text{--- (IV)}$$

$$\text{Similarly, for (II)} \rightarrow \frac{d\bar{b}}{ds} \cdot \bar{a} = -\sin \phi \frac{d\phi}{ds} \quad \text{--- (V)}$$

~~dividing~~ using eq (III), (IV) and (V)

$$\kappa(\bar{N} \cdot \bar{a}) = -\sin \theta \frac{d\theta}{ds} \quad \& \quad -\tau(\bar{N} \cdot \bar{a}) = -\sin \phi \frac{d\phi}{ds}$$

dividing above two equation

$$\frac{k(N \cdot \bar{a})}{-e(N \cdot \bar{a})} = \frac{\sin \theta \cdot d\theta}{\sin \phi \cdot d\phi}$$

$$\Rightarrow \boxed{\frac{\sin \theta \cdot d\theta}{\sin \phi \cdot d\phi} = -\frac{k}{e}}$$

Q.5 (c) Prove that $\nabla \times (\phi A) = (\nabla \phi) \times A + \phi (\nabla \times A)$

(10)

To prove: $\nabla \times (\phi A) = \nabla \phi \times A + \phi (\nabla \times A)$

we know that,

$$\nabla \times (\phi A) = \sum \hat{i} \times \frac{\partial}{\partial x} (\phi A)$$

$$= \sum \hat{i} \times \left(\frac{\partial \phi}{\partial x} A + \phi \frac{\partial A}{\partial x} \right)$$

$$= \sum (\hat{i} \times A) \frac{\partial \phi}{\partial x} + \sum \phi (\hat{i} \times \frac{\partial A}{\partial x})$$

$$= \left(\sum \hat{i} \frac{\partial \phi}{\partial x} \right) \times A + \phi \sum (\hat{i} \times \frac{\partial A}{\partial x})$$

$$= \nabla \phi \times A + \phi \nabla \times A$$

$$\therefore \left[\nabla \times A = \sum \hat{i} \times \frac{\partial A}{\partial x} \right]$$

$$\therefore \boxed{\nabla \times (\phi A) = \nabla \phi \times A + \phi (\nabla \times A)}$$

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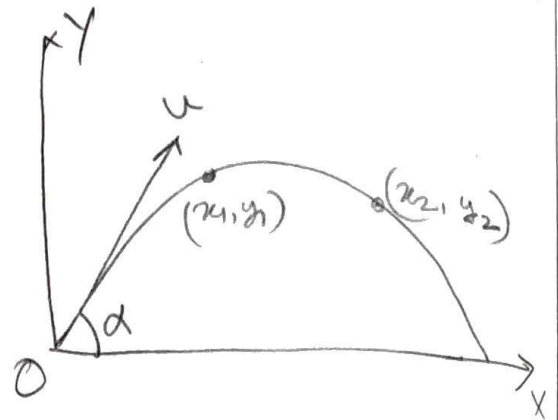
- Q.5 (d) A particle is projected in a direction making an angle α with the horizontal. It passes through the two points (x_1, y_1) and (x_2, y_2) . Prove that $\tan \alpha = \frac{y_1 R}{R x_1 - x_1^2} = \frac{x_2^2 y_1 - x_1^2 y_2}{(x_1 x_2)(x_2 - x_1)}$ where R denotes the horizontal range.

(10)

Let particle is projected with initial velocity u and angle α to horizontal from O .

Also, it passes through

$A(x_1, y_1)$ and $B(x_2, y_2)$



where OX is horizontal and OY is Vertical.

Let R be horizontal Range then $R = \frac{u^2 \sin 2\alpha}{g}$

Now, equation of projectile formed by particle is -

$$\begin{aligned} y &= x \tan \alpha - \frac{g}{2} \frac{x^2}{u^2 \cos^2 \alpha} \\ &= x \tan \alpha - \frac{g}{2 u^2 \cos \alpha} \cdot x^2 \cdot \sin \alpha \cdot \cos \alpha \\ &= x \tan \alpha - \frac{x^2}{R} \tan \alpha \end{aligned}$$

As, it passes through $A(x_1, y_1)$ & $B(x_2, y_2)$

$$y_1 = x_1 \tan \alpha - \frac{x_1^2 \tan \alpha}{R} \quad \text{--- (1)}$$

$$\Rightarrow \boxed{\frac{y_1 R}{R x_1 - x_1^2} = \tan \alpha}$$

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$$\& y_2 = x_2 \tan \alpha - \frac{x_2^2}{R} \tan \alpha \quad \text{--- (11)}$$

multiply (1) with x_2^2 and (11) with x_1^2

$$y_1 x_2^2 = x_1 x_2^2 \tan \alpha - \frac{x_1^2 x_2^2}{R} \tan \alpha \quad \text{and}$$

$$y_2 x_1^2 = x_1^2 x_2 \tan \alpha - \frac{x_1^2 x_2^2}{R} \tan \alpha$$

on subtracting above two eq^s $\rightarrow$

$$y_1 x_2^2 - y_2 x_1^2 = x_1 x_2 (x_2 - x_1) \tan \alpha$$

$$\Rightarrow \boxed{\tan \alpha = \frac{y_1 x_2^2 - x_1^2 y_2}{x_1 x_2 (x_2 - x_1)}}$$

$$\therefore \boxed{\tan \alpha = \frac{y_1 R}{R x_2 - x_1^2} = \frac{x_2^2 y_1 - x_1^2 y_2}{(x_1 x_2) (x_2 - x_1)}}$$

Ans.

Q.5 (e)

A cylinder of radius ' r ' whose axis is fixed horizontally, touches a vertical wall along a generating line. A flat beam of length ' l ' and weight ' W ' rests with its extremities in contact with the wall and the cylinder, making an angle 45° with the vertical. Prove that the reaction of the cylinder is $\frac{W\sqrt{5}}{2}$ and the pressure on the wall is $W/2$. Also prove that the ratio of radius of the cylinder to the length of the beam is $(5 + \sqrt{5}) : 4\sqrt{2}$.

Let cylinder of radius r with centre O touches wall at M . Rod AB of length l touches wall at A & cylinder at B then given that $\angle BAM = 45^\circ$ let R be reaction of cylinder on rod

N be reaction of wall to rod and W be weight of rod. Consider force diagram of rod.

$\therefore$ rod is in equilibrium,

$\therefore$ All 3 forces will pass through common point C .

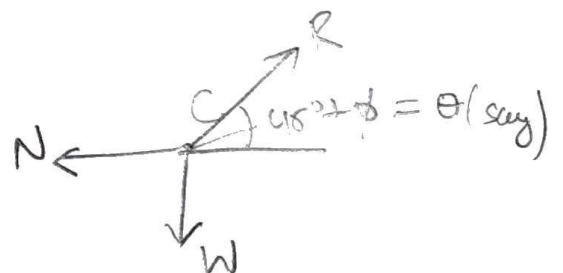
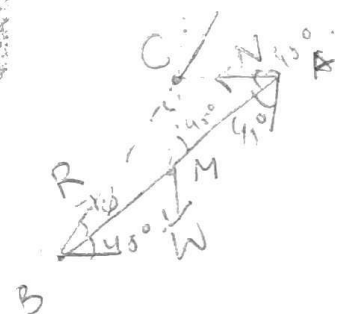
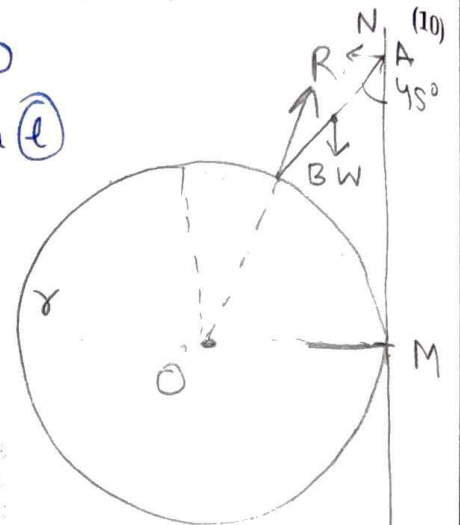
In $\triangle ACB$ and M is mid point, By m-n theorem

$$\left(\frac{l}{2} + \frac{l}{2}\right) \cot 45^\circ = \frac{1}{2} \cot \phi - \frac{1}{2} \cot 45^\circ$$

$$l = \frac{1}{2} \cot \phi - \frac{1}{2} \quad \cot \phi = 3$$

$$\Rightarrow \tan \phi = \frac{1}{3}$$

$$\text{Let } \theta = 45^\circ + \phi$$



NEXT IAS

$$\tan \theta = \frac{\tan 45^\circ + \tan \phi}{1 - \tan 45^\circ \tan \phi} = \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \approx 2$$

$$\Rightarrow \tan \phi = 2$$

Now, Lami's theorem at C $\rightarrow$

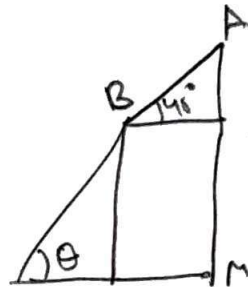
$$\frac{R}{\sin 90^\circ} = \frac{W}{\sin(\pi - \theta)} = \frac{N}{\sin(90^\circ + \theta)}$$

$$R = \frac{W}{\sin \theta} = \frac{N}{\cos \theta}$$

$$\therefore \tan \theta = 2 \quad \sin \theta = \frac{2}{\sqrt{5}} \quad \cos \theta = \frac{1}{\sqrt{5}}$$

$$\text{Now } R = \frac{W}{\sin \theta} = \frac{W\sqrt{5}}{2}$$

$$N = W \cot \theta = \frac{W}{2}$$



Now $\because OM = \text{radius}$

$$\Rightarrow OB \cos \theta + AB \cos 45^\circ = \text{radius}$$

Let R be radius of sphere.

$$\Rightarrow R \cos \theta + \frac{l}{\sqrt{2}} = R$$

$$\frac{R}{\sqrt{5}} + \frac{l}{\sqrt{2}} = R \Rightarrow \frac{R}{\sqrt{5}} \left(\frac{1}{\sqrt{5}} - 1 \right) = -\frac{l}{\sqrt{2}}$$

$$\frac{R}{l} = \frac{\sqrt{5}}{\sqrt{2}(\sqrt{5}-1)} = \frac{\sqrt{5}(\sqrt{5}+1)}{\sqrt{2}(5-1)}$$

$$\boxed{\frac{R}{l} = \frac{5+\sqrt{5}}{4\sqrt{2}}}$$

Q.6 (a) Reduce $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, where P, Q, R are functions of x , to normal form. Hence

solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2} \sin 2x$.

(20)

Given D.E. $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$. — (1)

Let $y = uv$ be general solution to eqⁿ (1)

then $\frac{dy}{dx} = \frac{du}{dx} v + u \frac{dv}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2} v + u \frac{d^2v}{dx^2} + 2 \frac{du}{dx} \frac{dv}{dx}$$

substituting in eq (1)

$$\frac{d^2u}{dx^2} \cdot v + u \frac{d^2v}{dx^2} + 2 \frac{du}{dx} \cdot \frac{dv}{dx} + P \cdot u \frac{dv}{dx} + Quv = R$$

$$+ P u \frac{dv}{dx} + Quv = R$$

making Coeff. of $\frac{dv}{dx} = 0$

$$2u' + Pu = 0$$

$$\Rightarrow \frac{u'}{u} = -\frac{P}{2} \Rightarrow \log u = -\int \frac{P}{2} dx$$

$$\Rightarrow \boxed{u = e^{\int \frac{P}{2} dx}}$$

$\therefore$ choosing $u = e^{\int \frac{P}{2} dx}$

$$\frac{d^2v}{dx^2} (u) + v \left(\frac{d^2u}{dx^2} + P \frac{du}{dx} + Qu \right) = R$$

$$\frac{d^2v}{dx^2} + \frac{\left(\frac{du}{dx^2} + P\frac{du}{dx} + Qu\right)}{u} v = R/u$$

Let $\frac{d^2v}{dx^2} + Iv = S$, $S = R/u$ and

where $I = \frac{1}{u} \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu \right)$

$$\frac{du}{dx} = e^{\int P/2 dx} \times (-P/2), \quad \frac{d^2u}{dx^2} = e^{\int P/2 dx} \left(\frac{P^2}{4} - \frac{P'}{2} \right)$$

$$\begin{aligned} \therefore I &= \frac{1}{u} \left(e^{\int P/2 dx} \left(\frac{P^2}{4} - \frac{P'}{2} \right) + e^{\int P/2 dx} \left(P(-P/2) + Qu \right) \right) \\ &= \frac{P^2}{4} - \frac{P'}{2} - \frac{P^2}{2} + Q \end{aligned}$$

$$\boxed{I = Q - \frac{P^2}{4} - \frac{P'}{2}} \quad \& \quad \boxed{S = R/u}$$

where $\frac{d^2v}{dx^2} + Iv = S$.

as required
Normal form

Now, Consider, $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2} \sin 2x$

In this, $P = -4x$ $Q = 4x^2 - 1$ $R = -3e^{x^2} \sin 2x$

On Converting to Normal form.

let $u = e^{\int P/2 dx} = e^{\int -2x dx} = e^{-x^2}$

$\Rightarrow \underline{\underline{u = e^{-x^2}}}$

$$\text{Then } I = Q - P' \frac{1}{4} - P' \frac{1}{2}$$

$$= 4x^2 - 1 - \frac{16x^2}{4} - \frac{(-4)}{2}$$

$$= 1$$

$$\& S = \frac{-3e^{x^2} \sin 2x}{e^{x^2}} = -3 \sin 2x$$

$\therefore$ Normal form of eqⁿ will be

$$\frac{d^2u}{dx^2} + u = -3 \sin 2x$$

It's Auxillary eqⁿ will be $\rightarrow$

$$m^2 + 1 = 0$$

$$\Rightarrow m = \pm i$$

$$\Rightarrow u_{c.s} = A \cos x + B \sin x$$

& for particular Integral

$$u_{p.i} = \frac{1}{D^2 + 1} (-3 \sin 2x) = \frac{-3}{-3} \sin 2x = \sin 2x$$

$$\therefore u_{g.s} = u_{c.s} + u_{p.i}$$

$$= A \cos x + B \sin x + \sin 2x$$

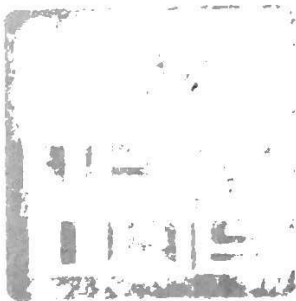
$\therefore$ general solⁿ for eq will be

$$y = u \times u_{g.s.}$$

$$y = e^{x^2} (A \cos x + B \sin x + \sin 2x)$$

Candidates
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NEXT IAS



Q.6 (b)

A particle of mass 5 units moves in a straight line towards a centre of force and the force varies inversely as the cube of distance. Starting from rest at the point A distant 20 units from centre of force O, it reaches a point B distant b from O. Find the time in reaching from A to B and the velocity at B. When will the particle reach at the centre? (15)

let particle of mass $m = 5$ unit
moves in straight toward centre O.
and Force $\propto \frac{1}{x^3}$, where x is
distance from O.

let,
At A, $a = 20$ unit

At B, $b = b$ unit

To find ① time ② from A to B

② velocity at B

③ ~~when~~ time when particle reaches O.

$$\therefore F \propto \frac{1}{x^3} \quad \text{let } F = \frac{-K}{x^3} \quad \text{--- ①} \quad k - \text{constant}$$

At any point P distance x from O,

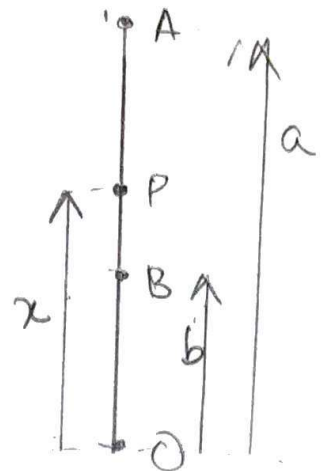
$$\text{acceleration} = \frac{d^2x}{dt^2} = \frac{F}{m} = \frac{-K}{5x^3} \quad \text{--- ②}$$

multiplying dx/dt on both side $\rightarrow$

$$(dx/dt) d(dx/dt) = \frac{-K}{5} \frac{1}{x^3} dx$$

$$\text{On integrating } \rightarrow \frac{1}{2} \left(\frac{dx}{dt} \right)^2 = \frac{-K}{5} \frac{1}{2(x+2)} + B$$

$$\text{at } x = a = 20 \quad \frac{dx}{dt} = 0$$



NEXT IAS

$$\Rightarrow B = + \frac{-k}{10(a^2)} = \frac{-k}{10(400)} = \frac{-k}{4000}$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = -\left(\frac{-k}{5x^2} \pm \frac{k}{2000}\right)$$

$$\boxed{\left(\frac{dx}{dt}\right)^2 = -\frac{k}{5} \left(\frac{1}{400} - \frac{1}{x^2}\right)} \quad \text{--- (iii)}$$

Now, $\frac{dx}{dt} = -\sqrt{\frac{-k}{5} \left(\frac{x^2-400}{400x^2}\right)}$ (—ve sign $\because$ x is in opposite direction of velocity).

$$\Rightarrow \frac{x dx}{\sqrt{x^2-400}} = -\sqrt{\frac{k}{2000}} dt$$

$$\sqrt{x^2-400} = -\sqrt{\frac{k}{2000}} t + C$$

at $t=0$ $x=20$, $\Rightarrow C=0$.

$$\Rightarrow \sqrt{x^2-400}$$

$$\frac{x dx}{\sqrt{400-x^2}} = -\sqrt{\frac{k}{2000}} dt$$

On integrating $\rightarrow$

$$-\frac{1}{2} \int \frac{-2x dx}{\sqrt{400-x^2}} = \int -\sqrt{\frac{k}{2000}} dt$$

$$\Rightarrow -\sqrt{400-x^2} = -\sqrt{\frac{k}{2000}} t + B$$

at $x=20, t=0 \Rightarrow B=0$

$$t = \sqrt{\frac{2000}{k}} \sqrt{400-x^2} \quad \text{--- (iv)}$$

at $x=b$

$$t = \sqrt{\frac{2000}{k}} \sqrt{400-b^2}$$

time from
A to B
Ans (1) #

from eq (ii) velocity at B $\rightarrow$

$$\left(\frac{dx}{dt}\right)^2 = \frac{k}{5} \left(\frac{1}{b^2} - \frac{1}{400}\right) \Rightarrow \frac{dx}{dt} = \sqrt{\frac{k}{5} \left(\frac{1}{b^2} - \frac{1}{400}\right)}$$

in direction towards O

Ans (i) #

Now, from eq (iv)

$$t = \sqrt{\frac{2000}{k}} \sqrt{400 - x^2} \quad \text{at } x = 0$$

$$t = \sqrt{\frac{2000}{k}} \sqrt{400}$$

time taken to reach at Centre = $400 \sqrt{5/k}$

Ans (ii) #

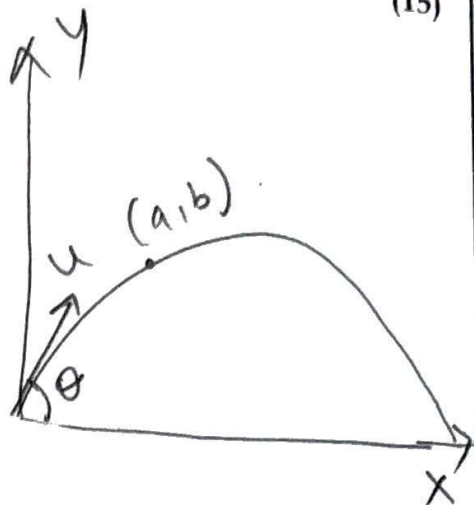


If α and β are the two possible directions to hit a given point (a, b) , then show that

$$\tan(\alpha + \beta) = -\frac{a}{b}$$

(15)

Let a particle is in projectile motion from origin O with initial velocity u and angle to horizontal Ox be θ



then, its path of projectile is given by

eqⁿ $\rightarrow$

$$y = x \tan \theta - \frac{g}{2u^2 \cos^2 \theta} x^2$$

if it passes through (a, b)

$$b = a \tan \theta - \frac{ga^2 \sec^2 \theta}{2u^2}$$

$$= a \tan \theta - \frac{ga^2 \tan^2 \theta}{2u^2} - \frac{ga^2}{2u^2}$$

$$\frac{ga^2 \tan^2 \theta}{2u^2} - a \tan \theta + b + \frac{ga^2}{2u^2} = 0$$

Now, this eqⁿ will give 2 values of

θ i.e. α & β .

then, $\tan \alpha + \tan \beta$ (Sum of roots) will be —

$$\tan \alpha + \tan \beta = \frac{a}{ga^2/2u^2}$$

(product of root)

$$\tan \alpha \cdot \tan \beta = \frac{b + ga^2/2u^2}{ga^2/2u^2}$$

Now, $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}$

$$= \frac{\frac{a}{ga^2/2u^2}}{\frac{(b + ga^2/2u^2)}{(ga^2/2u^2)}}$$

$$= \frac{a/(ga^2/2u^2)}{-b/(ga^2/2u^2)}$$

$$\boxed{\tan(\alpha + \beta) = -\frac{a}{b}}$$

Candidates
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Q.7 (a) Solve the differential equation: $(1-x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - (1+x^2)y = x$. (15)





NEXT IAS

Find the radii of curvature and torsion at a point of intersection of the surfaces

$$x^2 - y^2 = c^2, y = x \tanh\left(\frac{z}{c}\right).$$

(20)

Given curve: $x^2 - y^2 = c^2$ $y = x \tanh(z/c)$

To find radii of curvature & radius of torsion
($1/k$) ($1/\tau$)

Wkt, $k = \left| \frac{dz}{ds} \times \frac{d^2z}{ds^2} \right|$ & $\tau = \frac{\left[\frac{dz}{ds} \frac{d^2z}{ds^2} \frac{d^3z}{ds^3} \right]}{k^2}$

writing given curve in parameter form $\rightarrow$

let $x = c \cosh(t)$ $y = c \sinh(t)$
and $z = ct$

it satisfy given curve $\rightarrow x^2 - y^2 = c^2(\cosh^2 t - \sinh^2 t) = c^2$

any $y/x = \tanh(t) = \tanh(z/c)$

$\therefore x = c \cosh(t)$ $y = c \sinh(t)$ $z = ct$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = c \cosh(t)\hat{i} + c \sinh(t)\hat{j} + ct\hat{k} \quad \text{--- (I)}$

$\frac{d\vec{r}}{dt} = c \sinh(t)\hat{i} + c \cosh(t)\hat{j} + c\hat{k} \quad \text{--- (II)}$

$\left| \frac{d\vec{r}}{dt} \right| = \frac{ds}{dt} = \sqrt{c^2 + c^2 \sinh^2 t + c^2 \cosh^2 t}$
 $= c \cosh(t) \sqrt{2}$

$\therefore \frac{ds}{dt} = \sqrt{2} c \cosh(t) \quad \text{--- (III)}$

differentiating eq (1) w.r.t t

$$\frac{d^2 \vec{r}}{dt^2} = c \cosh(t) \hat{i} + c \sinh(t) \hat{j}$$

again diff. $\rightarrow$

$$\frac{d^3 \vec{r}}{dt^3} = c \sinh(t) \hat{i} + c \cosh(t) \hat{j}$$

$$\text{Now, } k = \left| \frac{d\vec{r}}{dt} \times \frac{d^2 \vec{r}}{dt^2} \right| / \left| \frac{d\vec{r}}{dt} \right|^3.$$

$$\begin{aligned} \frac{d\vec{r}}{dt} \times \frac{d^2 \vec{r}}{dt^2} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ c \sinh t & c \cosh t & c \\ c \cosh t & c \sinh t & 0 \end{vmatrix} \\ &= -c^2 \sinh t \hat{i} + c^2 \cosh t \hat{j} - c^2 \hat{k} \end{aligned}$$

$$\begin{aligned} \left| \frac{d\vec{r}}{dt} \times \frac{d^2 \vec{r}}{dt^2} \right| &= c^2 \sqrt{1 + \sinh^2 t + \cosh^2 t} \\ &= \sqrt{2} c^2 \cosh t \end{aligned}$$

$$\therefore k = \frac{\sqrt{2} c^2 \cosh t}{(\sqrt{2} c \cosh t)^3} = \frac{\sqrt{2} c^2 \cosh t}{2\sqrt{2} c^3 \cosh^3 t}$$

$$k = \frac{1}{2 c \cosh^2 t}$$

$$\therefore \text{Radius of curvature} = \frac{1}{k} = 2c \cosh^2(t)$$

$$\boxed{\text{Radius of Curvature} = 2c \cosh^2(z/c)}$$

Ans (1)

NEXT IAS

$$\text{Now, } \tau = \frac{d^3 \vec{r}}{dt^3} \cdot \left(\frac{d\vec{r}}{dt} \times \frac{d^2 \vec{r}}{dt^2} \right) \div \left| \frac{d\vec{r}}{dt} \times \frac{d^2 \vec{r}}{dt^2} \right|^2$$

$$\tau = \left(c \sinh(t) \hat{i} + c \cosh(t) \hat{j} \right) \cdot \begin{pmatrix} -c^2 \sinh t \hat{i} \\ + c^2 \cosh t \hat{j} \\ - c^2 \hat{k} \end{pmatrix}$$

$$= \frac{(\sqrt{2} c^2 \cosh t)^2}{-c^3 \sinh^2 t + c^3 \cosh^2 t}$$

$$= \frac{2 c^4 \cosh^2 t}{2 c^4 \cosh^2 t} = \frac{1}{2 c \cosh^2 t}$$

$$\therefore \text{radius of torsion} = \frac{1}{c}$$

$$= 2 c \cosh^2(t)$$

$$\boxed{\text{radius of torsion} = 2 c \cosh^2(z/c)}$$

Ans (11)



NEXT IAS

Find the law of force for the orbit $r^2 = a^2 \cos 2\theta$ (the pole being the centre of the force). (15)

Given orbit: $r^2 = a^2 \cos 2\theta$

Let $u = \frac{1}{r} \Rightarrow \frac{1}{u^2} = \frac{1}{a^2} \sec 2\theta$ — (1)

We know that law of force is given by $\rightarrow$

$$\frac{P}{h^2 u^2} = u + \frac{d^2 u}{d\theta^2} \quad \text{--- (11)}$$

diff. eqⁿ (1) w.r.t. θ

$$-\frac{2}{u^3} \cdot \frac{du}{d\theta} = \frac{1}{a^2} \cdot 2 \sec 2\theta \cdot \tan 2\theta$$

$$\frac{du}{d\theta} = -\frac{u^3}{a^2} (\sec 2\theta \cdot \tan 2\theta)$$

$$= -\frac{u}{a^2} \cdot a^2 \cdot \tan 2\theta$$

$$= -u \tan 2\theta$$

again diff. w.r.t. $\theta \rightarrow$

$$\frac{d^2 u}{d\theta^2} = -\frac{du}{d\theta} \tan 2\theta - 2u \sec^2 2\theta$$

$$= -\tan 2\theta (-u \tan 2\theta) - 2u \sec^2 2\theta$$

$$= u \tan^2 2\theta - 2u \sec^2 2\theta$$

$$= u (\tan^2 2\theta - 2 \sec^2 2\theta)$$

diff. eqn wrt $\theta \rightarrow$

$$2u \frac{du}{d\theta} = \frac{2 \cdot \sec 2\theta \cdot \tan 2\theta}{a^2}$$

$$u \frac{du}{d\theta} = \frac{\sec 2\theta \tan 2\theta}{a^2}$$

Again diff. wrt θ

$$\left(\frac{du}{d\theta}\right)^2 + u \frac{du}{d\theta^2} = \left(\frac{\sec 2\theta \tan^2 2\theta + \sec^3 2\theta}{a^2}\right) \times 2$$

$$u \frac{d^2u}{d\theta^2} = \frac{2 \cdot \sec 2\theta}{a^2} (1 + 2 \tan^2 2\theta) - \frac{\sec^2 2\theta \cdot \tan^2 2\theta}{u^2 a^4}$$

$$u \frac{d^2u}{d\theta^2} = \cancel{\sec 2\theta} 2u^2 (1 + 2 \tan^2 2\theta) - \frac{\sec 2\theta \tan^2 2\theta}{a^2}$$

$$= u^2 (2(1 + 2 \tan^2 2\theta) - \tan^2 2\theta) \quad \left(\text{from eqn (1)}\right)$$

$$= u^2 (2 + 3 \tan^2 2\theta)$$

$$\frac{d^2u}{d\theta^2} = u (2 + 3 \tan^2 2\theta)$$

$\therefore$ putting in eqn (11)

$$P = h^2 u^2 \left(u + \frac{d^2u}{d\theta^2} \right)$$

$$= h^2 u^2 (u) (3) (1 + \tan^2 2\theta)$$

$$= 3h^2 u^3 (1 + \tan^2 2\theta)$$

$$P = 3h^2 u^3 (\sec^2 2\theta)$$

$$P = 3h^2 u^3 (u^2 a^2)^2 \quad (\text{from eqn (1)})$$

$$= 3h^2 a^4 u^7$$

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NEXT IAS

$$P = \frac{3k^2 a^4}{27}$$

$$\Rightarrow \boxed{P \propto 1/27}$$

