

NEXT IAS

(To be filled by candidate)

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 Roll No.: MT 24 MATL 1011 Date of Examination: 15/7
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 Exam Centre: Old Rajinder Nagar ☒ Online ☒

CSE (MAINS) TEST SERIES-2024

Test-03

**MATHEMATICS
OPTIONAL**

Dated : 07-07-2024

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are EIGHT questions divided in Two sections and printed in ENGLISH.

Candidate has to attempt FIVE questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

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Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	20			
2. (c)	21	15			
3. (a)	24	20			
3. (b)	28	15			
3. (c)	31	15			
4. (a)	34	15			
4. (b)	37	20			
4. (c)	41	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	20			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	81	20			
Grand Total					

Remarks

Observations:

Given equation is quadratic in y

$$(y - \frac{1}{2})^2 = \frac{1}{4} \Rightarrow y = \frac{1}{2} \pm \frac{1}{2}$$

$$y = \frac{1}{2} \pm \frac{1}{2} \Rightarrow y = 1 \text{ or } 0$$

$$0 = \left(\frac{y - \frac{1}{2}}{\frac{1}{2}} \right) \left(\frac{y - \frac{1}{2}}{\frac{1}{2}} \right) \Rightarrow y = 1 \text{ or } 0$$

$$D = (y - \frac{1}{2})^2 - \left(\frac{1}{2} \right)^2 = 0$$

for solution

$$D = 0 \Rightarrow y = 1 \text{ or } 0$$

$$0 = (y - \frac{1}{2})^2 - \left(\frac{1}{2} \right)^2 \Rightarrow y = 1 \text{ or } 0$$

$$0 = (y - \frac{1}{2})^2 - \left(\frac{1}{2} \right)^2 \Rightarrow y = 1 \text{ or } 0$$

$$y = 1 \text{ or } 0$$

Let $y = 1$ and $y = 0$

Let $y = 1$ and $y = 0$

SECTION-A

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(10)

Q.1 (a)

Find the singular solution of the differential equation $9p^2(2-y)^2 = 4(3-y)$.Given equation is solvable for p

$$\Rightarrow 9p^2 = \frac{4(3-y)}{(2-y)^2}$$

$$p = \pm \frac{2}{3} \sqrt{\frac{3-y}{(2-y)^2}} = \pm \frac{2\sqrt{3-y}}{3(2-y)}$$

$$\left(p - \frac{2\sqrt{3-y}}{3(2-y)} \right) \left(p + \frac{2\sqrt{3-y}}{3(2-y)} \right) = 0$$

$$9p^2(2-y)^2 - 4(3-y) = 0$$

for singular solution

$$D = B^2 - 4AC = 0$$

$$\Rightarrow 0^2 - 4(9)(2-y)^2(3-y) = 0$$

$$\Rightarrow (2-y)^2(3-y) = 0$$

$$\Rightarrow y = 2, 3$$

 $y = 2$ not satisfy and $y = 3$

satisfy given equation

$\therefore y = 3$ is required singular
solution



Q.1 (b)

Using Laplace Transform, solve the IVP $ty'' + 2ty' + 2y = 2$ and $y(0)$ is arbitrary. (10)

Does this problem has a unique solution.

Formula used

$$L(y'(t)) = sy(s) - y(0)$$

$$L(y''(t)) = s^2y(s) - sy(0) - y'(0)$$

$$L\left(\frac{y}{t}\right) = \int_0^\infty y(s) ds$$

$$L(ty) = -\frac{d}{ds}y(s)$$

$$ty'' + 2ty' + 2y = 2$$

$$y'' + 2y' + \frac{2y}{t} = \frac{2}{t}$$

Taking Laplace transform both sides

$$s^2y(s) - sy(0) - y'(0) + 2sy(s) - 2y(0)$$

$$+ 2 \int_0^\infty y(s) ds = \text{bad}$$

$$-\frac{d}{ds}(s^2y(s) - sy(0) - y'(0)) + 2 \frac{d}{ds}(sy(s) - y(0))$$

$$+ 2y(s) = \frac{2}{s}$$

$$-2sy'(s) + y(0) - 2sy'(s) + 2y(s) = \frac{2}{s}$$

$$-y'(s)[s^2+2s] - 2sy(s) + 1 = \frac{2}{s}$$

$$2sy(s) = 1 - \frac{2}{s} = \frac{s-2}{s}$$

$$2y(s) = \frac{s-2}{s^2} = \frac{1}{s} - \frac{2}{s^2}$$

Taking inverse Laplace transform

$$2y(t) = 1 - \frac{1}{t}$$

$$y'(s) + \frac{2y(s)}{s+2} = \frac{s-2}{s^2(s+2)}$$

$$\text{Integrating factor} = e^{\int \frac{2}{s+2} ds} = (s+2)^2$$

$$[y(s)][s+2]^2 = \int \frac{(s-2)(s+2)}{s^2} ds$$

$$= \int 1 - \frac{4}{s^2} ds = s + \frac{4}{s} + C_1$$

$$y(s) = \frac{s}{(s+2)^2} + \frac{4}{s(s+2)^2} + \frac{C_1}{(s+2)^2} = \frac{1}{s} + \frac{C_1-4}{(s+2)^2}$$

$$y(t) = 1 + (C_1-4)t e^{-2t}, \quad y(0) = 1 \Rightarrow C_1 = 4$$

C_1 is any real constant. Non-unique sol

Q.1 (c)

Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^2)dx - 3x^2y^2dy$ along the path $x^4 - 6xy^2 = 4y^2$.

(10)

$$\int Mdx + Ndy$$

$$M = 10x^4 - 2xy^2; N = -3x^2y^2$$

$$\frac{\partial M}{\partial y} = -6xy^2; \frac{\partial N}{\partial x} = -6xy^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ given equation is exact

$\therefore$ Integration is path independent

$$\int Mdx + Ndy = \int_{y \text{ constant}} Mdx + \int_{\text{term not containing } x} Ndy$$

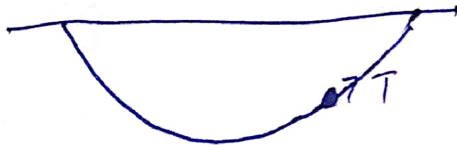
$$\left[(2x^5 - x^2y^3) + 0 \right]_{(0,0)}^{(2,1)}$$

$$2^6 - 2^2 = 2^2(2^4 - 1) = 60$$

Q.1 (d)

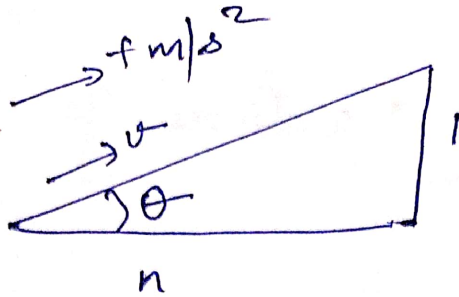
If mass density at any point of a cord varies as the radius of curvature of the curve in which it hangs freely under gravity, then prove that this curve is the catenary of uniform strength. (10)

$$\text{Radius of curvature} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)}$$



Q.1 (e)

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 A train of mass M kg is ascending a smooth incline of 1 in n and velocity is v m/s, its acceleration is f m/s². Prove that the effective power of the engine is $Mv(nf + g)/n$ watts. (10)



$$\tan \theta = \frac{1}{n}$$

$$Mf = F - Mg \sin \theta$$

Work
Power =

$$F(vt)$$

$$\text{Power} = \frac{\text{Work}}{t} = Fv$$

$$F = M(g \sin \theta + f)$$

$$= M(g \sqrt{1 - \cos^2 \theta} + f)$$

$$= M(g \sqrt{1 - \frac{1}{\sec^2 \theta}} + f)$$

$$= M(g \sqrt{1 - \frac{1}{1 + \tan^2 \theta}} + f)$$

$$= M(g \sqrt{1 - \frac{1}{1 + \frac{1}{n^2}}} + f)$$

$$= M(g \sqrt{\frac{\frac{1}{n^2}}{1 + \frac{1}{n^2}}} + f)$$

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$$F = M \left(g \sqrt{\frac{1}{1+n^2}} + f \right)$$

$$P = Fv = Mv \left(f + g \sqrt{\frac{1}{1+n^2}} \right)$$



Q.2 (a)

Show that the equation $(4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$ represents family of hyperbolas having as asymptotes the lines $x + y = 0$, $2x + y + 1 = 0$. (15)

Given $(4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$

can be written as

$$Mdx + Ndy = 0$$

$$\frac{\partial M}{\partial y} = 3 ; \quad \frac{\partial N}{\partial x} = 3$$

$\Rightarrow$ It is exact differential equation

as $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\therefore \int_{y \text{ constant}} M dx + \int_{\text{terms not involving } x} N dy = C$$

$$\int_{y \text{ constant}} 4x + 3y + 1 + \int 2y + 1 dy = C$$

$$2x^2 + x(3y + 1) + y^2 + y = C$$

$$x(2x + y + 1) + y(2x + y + 1) = C$$

$$(2x + y + 1)(x + y) = C$$

Now

$$2x^2 + y^2 + 3xy + x + y = C$$

$$ax^2 + by^2 + 2hxy + ux + vy = C$$

$$h^2 - ab = \left(\frac{3}{2}\right)^2 - 2 \cdot \frac{1}{8} > 0$$

$\Rightarrow$ It represents hyperbola

Also $(2x + y + 1)(x + y) = C$

$$\therefore 2x + y + 1 = 0 \text{ and}$$

$$x + y = 0 \text{ are}$$

its asymptotes.

b) Find the values of a , b and c such that the maximum value of directional derivative of $f = axy^2 + byz + cx^2z^2$ at $(1, -1, 1)$ is in the direction parallel to y axis and has magnitude 6. (20)

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \\ &= (ay^2 + 2cxz^2) \hat{i} + (2axy + bz + 2cxz) \hat{j} \\ &\quad + (by + 2cx^2z) \hat{k}\end{aligned}$$

$$\begin{aligned}\nabla f(1, -1, 1) &= (a + 2c) \hat{i} + (-2a + b + 2c) \hat{j} \\ &\quad + (-b + 2c) \hat{k}\end{aligned}$$

$$|\nabla f|_{(1, -1, 1)} = 6$$

Also direction is given as $\hat{j}$

We know ~~the~~ directional derivative is maximum in direction of

$$\frac{\nabla f}{|\nabla f|}$$

$$\therefore \frac{\nabla f}{|\nabla f|}(1, -1, 1) = \hat{j}$$

~~the~~ comparing

$$a + 2c = 0 = 2c - b \Rightarrow b = -a$$

$$\frac{-2a + b}{|\nabla f|} = 1$$

$$\frac{-3a}{|Df|} = 1$$

Given $Df = 6$

$$\Rightarrow a = -2$$

$$b = -a = 2$$

$$2c + a = 0$$

$$\Rightarrow c = 1$$

Ans

$$a = -2$$

$$b = 2$$

$$c = 1$$

$$d = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}$$

$$f(x) = (1, 1, 1, \frac{1}{\sqrt{2}})$$

$$a + b + c + d = 0 \Rightarrow -2 + 2 + 1 + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}}$$

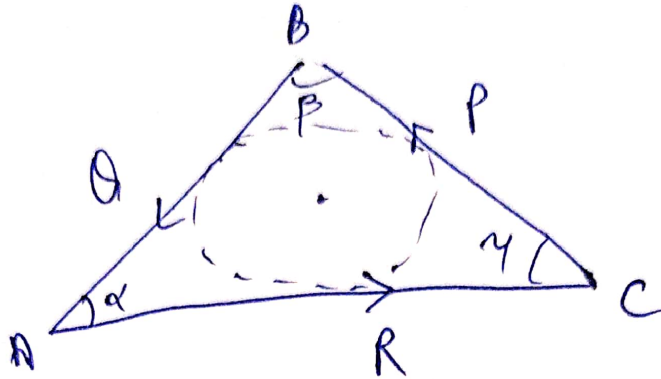
$$a + b + c + d = 0 \Rightarrow -2 + 2 + 1 + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}}$$

(c)

Three forces P , Q and R act along the sides BC , CA and AB of $\triangle ABC$ in order to keep the system in equilibrium. If the resultant force touches the inscribed circle, then prove that

$$\frac{1 + \cos \alpha}{P} + \frac{(1 + \cos \beta)}{Q} + \frac{1 + \cos \gamma}{R} = 0.$$

(15)



Q.4 (a)

Given that $y_1 = x^2$ is a solution of the differential equation, $x^3 \frac{d^2 y}{dx^2} - (x^2 + 3x) \frac{dy}{dx} + 6y = 0$, find the other linearly independent solution of the above differential equation and write down the general solution of the differential equation. (15)

$$x^3 \frac{d^2 y}{dx^2} - (x^2 + 3x) \frac{dy}{dx} + 6y = 0$$

$$\frac{d^2 y}{dx^2} - \left(\frac{1}{x} + \frac{3}{x^2} \right) \frac{dy}{dx} + \frac{6y}{x^3} = 0$$

let $y = uv$ is solution of given differential equation

Then we know that

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R$$

$$\frac{d^2 v}{dx^2} + \left(P + \frac{2}{u} \frac{du}{dx} \right) \frac{dv}{dx} = \frac{R}{u}$$

where u is a solution of

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = 0$$

$$\text{Here } P = -\left(\frac{1}{x} + \frac{3}{x^2} \right)$$

$$Q = \frac{6}{x^3}$$

$$\frac{d^2 v}{dx^2} + \left(-\frac{1}{x} - \frac{3}{x^2} + \frac{2}{x^2} (2x) \right) \frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + 3\left(\frac{1}{x} - \frac{1}{x^2}\right)\frac{dv}{dx} = 0$$

$$\text{let } \frac{dv}{dx} = t$$

$$\frac{dt}{dx} + 3\left(\frac{1}{x} - \frac{1}{x^2}\right)t = 0$$

$$\frac{dt}{t} = 3\left(\frac{1}{x^2} - \frac{1}{x}\right)dx$$

Integrating

$$\log t_1 = 3\left[-\frac{1}{x} - \log x\right]$$

$$t_1 = e^{-3\left(\frac{1}{x} + \log x\right)} = e^{-3/x} \cdot e^{-3 \log x}$$

$$c_1 \frac{dv}{dx} = \frac{e^{-3/x}}{x^3}$$

$$c_1 dv = dx \frac{e^{-3/x}}{x^3}$$

~~$$\text{let } e^{-3/x}$$~~

$$c_1 \int dv = \int \frac{e^{-3/x}}{x^3} dx$$

$$e^{-1/x} = \alpha$$

$$-\frac{1}{x^2} dx = d\alpha$$

$$c_1 \int dv = -\int \alpha e^{-3\alpha} d\alpha$$

$$c_1 \int dx = - \int x e^{-3x} dx$$

$$\begin{aligned} c_1 v + c_2 &= - \left[\frac{e^{-3x}}{(-3)} \frac{x^2}{2} - \int \frac{e^{-3x}}{(-3)} \right] \\ &= - \left[\frac{e^{-3x} x}{(-3)} - \frac{e^{-3x}}{9} \right] \\ &= \frac{e^{-3x}}{3} \left[x - \frac{1}{3} \right] \end{aligned}$$

$$c_1 v + c_2 = \frac{e^{-3/x}}{3} \left[\frac{1}{x} - \frac{1}{3} \right]$$

$$v = c_3 \left[\frac{e^{-3/x}}{3} \left(\frac{1}{x} - \frac{1}{3} \right) - c_2 \right] \text{ where } c_3 = \frac{1}{c_1}$$

Now

$$y = uv = c_3 \left[\frac{e^{-3/x}}{3} \left(x - \frac{x^2}{3} \right) - c_2 x^2 \right]$$

c_3, c_2 are arbitrary constant

$$y = c_3 \frac{e^{-3/x}}{3} \left(x - \frac{x^2}{3} \right) + c_4 x^2$$

where

$$c_4 = -c_2 c_3$$

an arbitrary constant

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- 4 (b) Let $F = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$. (a) Calculate $\nabla \times F$. (b) Evaluate $\oint F \cdot d\mathbf{r}$ around any closed path and explain the results. (20)

a) $\nabla \times F \Rightarrow$

$$F = \frac{-y}{x^2 + y^2} \hat{i} + \frac{x}{x^2 + y^2} \hat{j}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

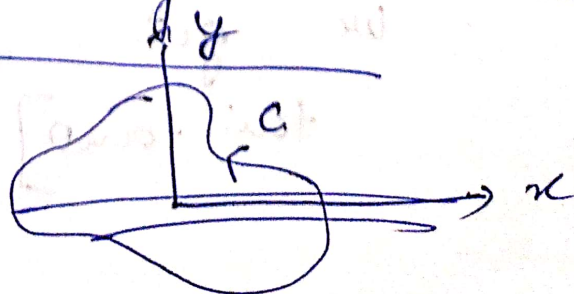
$$= \hat{k} \left(\frac{\partial}{\partial x} \frac{x}{x^2 + y^2} + \frac{\partial}{\partial y} \frac{y}{x^2 + y^2} \right)$$

$$= \hat{k} \left(\frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} + \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} \right)$$

$$= \hat{k} \begin{bmatrix} 0 \end{bmatrix} = \textcircled{0}$$

b) $\oint F \cdot d\mathbf{r}$

~~Case I when origin is inside the path.~~

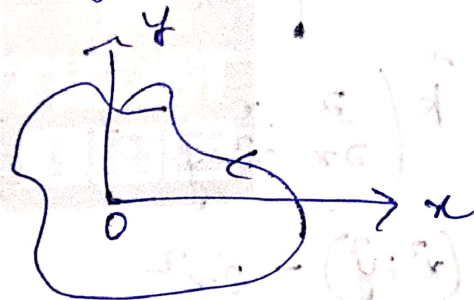


$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \frac{-y \cdot dx + x \cdot dy}{x^2 + y^2}$$

$$= \oint_C \frac{x dy - y dx}{x^2 + y^2}$$

$$= \oint_C d \tan^{-1} \left(\frac{y}{x} \right)$$

Case I When origin is inside the path



$$\oint_C \vec{F} \cdot d\vec{r} = \int_{\theta=0}^{2\pi} d \tan^{-1} \left(\frac{y}{x} \right)$$

$$= \left[\tan^{-1} \frac{y}{x} \right]_{\theta=0}^{2\pi}$$

Let $y = r \sin \theta$

$x = r \cos \theta$

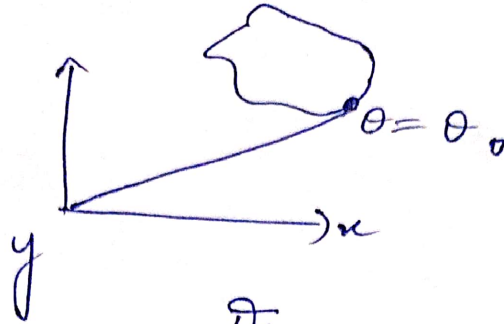
We get

$$\left[\tan^{-1} \tan \theta \right]_{\theta=0}^{2\pi} = 2\pi - 0$$

$$= 2\pi$$

Case II
path

when origin is outside the



$$\oint F \cdot dr = \int_{\theta=\theta_0}^{\theta_0} d \tan^{-1}\left(\frac{y}{x}\right)$$

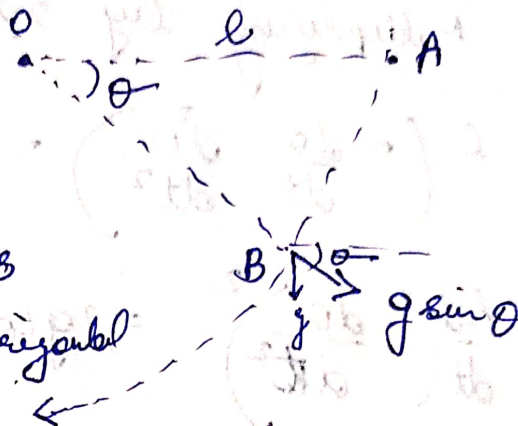
$$= \left[\tan^{-1}\left(\frac{y}{x}\right) \right]_{\theta=\theta_0}^{\theta_0} = \theta_0 - \theta_0 = 0.$$

2.4 (c)

A particle attached to a fixed peg O by a string of length l , is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with the horizontal, the resultant acceleration is $g\sqrt{1+3\sin^2\theta}$.

(15)

Let particle start from A and reach B when the string makes angle θ with OA (horizontal)



The components of $\vec{g}$ is radial direction is $g \sin \theta$.

The Tension is T at θ position

The radius is length of string.

As particle is dropped therefore initial velocity is 0

$$\text{Radial acceleration} = \frac{v^2}{r}$$

$$\text{Transverse acceleration} = r \frac{d^2\theta}{dt^2}$$

$$\text{here } r = l$$

$$T - mg \sin \theta = \frac{mv^2}{r}$$

$$m \frac{dv^2}{dt^2} = mg \cos \theta \Rightarrow l \frac{d^2\theta}{dt^2} = g \cos \theta$$

$$l \frac{d^2\theta}{dt^2} = g \cos \theta$$

Multiplying by $2 \frac{d\theta}{dt}$

$$l \left(2 \frac{d\theta}{dt} \frac{d^2\theta}{dt^2} \right) = 2g \cos \theta \frac{d\theta}{dt}$$

$$l \frac{d}{dt} \left(\frac{d\theta}{dt} \right)^2 = 2g \cos \theta \frac{d\theta}{dt}$$

Integrating

$$l \left(\frac{d\theta}{dt} \right)^2 = 2g \sin \theta + C_1$$

When $\theta = 0$; $\frac{d\theta}{dt} = 0 \Rightarrow C_1 = 0$

$$\Rightarrow \frac{d\theta}{dt} = \pm \sqrt{\frac{2g \sin \theta}{l}}$$

($\because \theta$ increases with t)

$$v = l \frac{d\theta}{dt}$$

$$\frac{v^2}{l} = \frac{1}{l} \left(l \frac{d\theta}{dt} \right)^2 = 2g \frac{\sin \theta}{l}$$

$$l \frac{d^2\theta}{dt^2} = g \cos \theta$$

Resultant acceleration

$$= \sqrt{(\text{radial})^2 + (\text{transverse})^2}$$

$$= \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(2 \frac{d^2\theta}{dt^2}\right)^2}$$

$$= \sqrt{4g^2 \sin^2 \theta + g^2 \cos^2 \theta}$$

$$= \sqrt{3g^2 \sin^2 \theta + g^2}$$

$$\left(\cos^2 \theta + \sin^2 \theta = 1 \right)$$

$$= g \sqrt{1 + 3 \sin^2 \theta}$$

SECTION-B

Q.5 (a) Solve the initial value problem $(2x^2 + y)dx + (x^2y - x)dy = 0$, $y(1) = 2$.

(10)

$$Mdx + Ndy = 0$$

$$\frac{\partial M}{\partial y} = 1 \quad ; \quad \frac{\partial N}{\partial x} = 2xy - 1$$

$$(2x^2)dx + x^2y dy = xdy - ydx$$

$$2dx + ydy = \frac{xdy - ydx}{x^2}$$

$$d(2x) + d\left(\frac{y^2}{2}\right) = d\left(\frac{y}{x}\right)$$

Integrating

$$2x + \frac{y^2}{2} = \frac{y}{x} + C$$

$$y(1) = 2$$

$$2 + \frac{2^2}{2} = \frac{2}{1} + C$$

$$C = 2$$

$$\Rightarrow 2x + \frac{y^2}{2} = \frac{y}{x} + 2$$

Q.5 (b) If the unit vector $\bar{t}$ and binormal $\bar{b}$ makes angle θ and ϕ respectively with a constant unit vector $\bar{a}$, prove that $\frac{\sin \theta}{\sin \phi} \frac{d\theta}{d\phi} = -\frac{\kappa}{\tau}$. (10)

$$\bar{t} \cdot \bar{a} = \cos \theta \quad \text{--- i}$$

$$\bar{b} \cdot \bar{a} = \sin \phi \quad \text{--- ii}$$

$$|\bar{a}| = |\bar{t}| = |\bar{b}| = 1$$

Differentiating both sides of
 (i), (ii) w.r.t parameter α of
 curve

$$\frac{d(\bar{a} \cdot \bar{t})}{d\alpha} = \frac{d\bar{a}}{d\alpha} \cdot \bar{t} + \bar{a} \cdot \frac{d\bar{t}}{d\alpha}$$

$$= 0 + \bar{a} \cdot \frac{d\bar{t}}{d\alpha}$$

$$\text{RHS} = \frac{d(\cos \theta)}{d\alpha} = \frac{d \cos \theta}{d\theta} \frac{d\theta}{d\alpha}$$

$$\text{i.e.} \quad \bar{a} \cdot \frac{d\bar{t}}{d\alpha} = -\sin \theta \frac{d\theta}{d\alpha} \quad \text{--- iii}$$

similarly for (ii)

$$\bar{a} \cdot \frac{d\bar{b}}{d\alpha} = \sin \phi \frac{d\phi}{d\alpha} \quad \text{--- iv}$$

Dividing iii and iv

$$\frac{\bar{a} \cdot \frac{d\bar{t}}{d\alpha}}{\bar{a} \cdot \frac{d\bar{b}}{d\alpha}} = \frac{-\sin \theta \frac{d\theta}{d\alpha}}{\sin \phi \frac{d\phi}{d\alpha}}$$

Divide by ds , s is arc length

$$\frac{\bar{a} \cdot \frac{d\bar{t}}{ds}}{\bar{a} \cdot \frac{d\bar{b}}{ds}} = \frac{\sin \theta}{\sin \phi} \frac{d\theta}{d\phi}$$

We know

$$NK = \frac{d\bar{t}}{ds} - \frac{d\bar{b}}{ds} = -ZN$$

$$\frac{\bar{a} \cdot (NK)}{\bar{a} \cdot (-ZN)} = \frac{\sin \theta}{\sin \phi} \frac{d\theta}{d\phi}$$

$$\frac{\sin \theta}{\sin \phi} \frac{d\theta}{d\phi} = -\frac{K}{Z}$$

Q.5 (c)

Prove that $\nabla \times (\phi A) = (\nabla \phi) \times A + \phi (\nabla \times A)$

(10)

$$A = \sum A_z \hat{i}$$

$$\nabla \times (\phi A)$$

$$= \sum \hat{k} \left(\frac{\partial \phi A_y}{\partial x} - \frac{\partial \phi A_x}{\partial y} \right)$$

$$= \sum \hat{k} \left(A_y \frac{\partial \phi}{\partial x} + \phi \frac{\partial A_y}{\partial x} - \phi \frac{\partial A_x}{\partial y} - A_x \frac{\partial \phi}{\partial y} \right)$$

$$= \sum \hat{k} (\phi) \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + \hat{k} \left(A_y \frac{\partial \phi}{\partial x} - A_x \frac{\partial \phi}{\partial y} \right)$$

$$= \sum \phi \left[\hat{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] + \sum \hat{k} \left(A_y \frac{\partial \phi}{\partial x} - A_x \frac{\partial \phi}{\partial y} \right)$$

$$= \phi \sum \hat{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + \sum \hat{k} \left(A_y \frac{\partial \phi}{\partial x} - A_x \frac{\partial \phi}{\partial y} \right)$$

$$= \phi (\nabla \times A) + (\nabla \phi) \times A$$

To show $(\nabla \phi) \times A = \sum \hat{k} \left(A_y \frac{\partial \phi}{\partial x} - A_x \frac{\partial \phi}{\partial y} \right)$

$$\nabla \phi = \sum \frac{\partial \phi}{\partial x} \hat{i}$$

$$A = \sum A_z \hat{i}$$

$$\nabla \phi \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\hat{z}^R \left(A_y \frac{\partial \phi}{\partial x} - A_x \frac{\partial \phi}{\partial y} \right)$$

Hence proved



Q.5 (d)

A particle is projected in a direction making an angle α with the horizontal. It passes through the two points (x_1, y_1) and (x_2, y_2) . Prove that $\tan \alpha = \frac{y_1 R}{R x_1 - x_1^2} = \frac{x_2^2 y_1 - x_1^2 y_2}{(x_1 x_2)(x_2 - x_1)}$ (10)

where R denotes the horizontal range.

We know trajectory equation is

$$y = x \tan \alpha - \frac{g x^2}{2u^2 \cos^2 \alpha}$$

$$y_1 = x_1 \tan \alpha - \frac{g x_1^2}{2u^2 \cos^2 \alpha} \quad \text{--- (i)}$$

$$y_2 = x_2 \tan \alpha - \frac{g x_2^2}{2u^2 \cos^2 \alpha} \quad \text{--- (ii)}$$

~~$y = x \tan \alpha - \frac{g x^2}{2u^2 \cos^2 \alpha}$~~ Subtracting

$$(y_1 - y_2) = \tan \alpha (x_1 - x_2) + \frac{g}{2u^2 \cos^2 \alpha} (x_2^2 - x_1^2)$$

$$\frac{y_1 - y_2}{x_1 - x_2} = \tan \alpha + \frac{g(x_2 + x_1)}{2u^2 \cos^2 \alpha}$$

Eliminating $\frac{g}{2u^2 \cos^2 \alpha}$ from (i) & (ii)

$$\frac{y_1 - x_1 \tan \alpha}{y_2 - x_2 \tan \alpha} = \frac{x_1^2}{x_2^2}$$

$$\tan \alpha (x_2 x_1^2 - x_1^2 x_2) = y_2 x_1^2 - y_1 x_2^2$$

$$\tan \alpha = \frac{x_2^2 y_1 - x_1^2 y_2}{x_1 x_2 (x_2 - x_1)}$$

From

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Candidates must not write on this margin

$$y_1 = x_1 \tan \alpha - \frac{gx_1^2}{2u^2} (1 + \tan^2 \alpha)$$

$$\tan \alpha = \frac{-x_1 \pm \sqrt{x_1^2 - 4\left(-\frac{gx_1^2}{2u^2}\right)\left(-\frac{gx_1^2}{2u^2} - y_1\right)}}{2\left(-\frac{gx_1^2}{2u^2}\right)}$$

As projectile is projected up from ground $\tan \alpha < 0$ is assured, Hence taking the sign

$$\tan \alpha = \frac{-1 + \sqrt{1 + \frac{2g}{u^2} \left(\frac{gx_1^2}{2u^2} + y_1\right)}}{-\frac{gx_1}{2u^2}}$$

$$y_1 = x_1 \tan \alpha - \frac{gx_1^2}{2u^2 \cos^2 \alpha} = x_1 \tan \alpha - \frac{gx_1^2}{2u^2 \cos^2 \alpha}$$

$$R = \frac{u^2 \sin 2\alpha}{g}$$

$$y_1 = x_1 \tan \alpha \left[1 - \frac{gx_1}{2u^2 \tan \alpha \cos^2 \alpha} \right]$$

$$y_1 = x_1 \tan \alpha \left[1 - \frac{x_1}{R} \right]$$

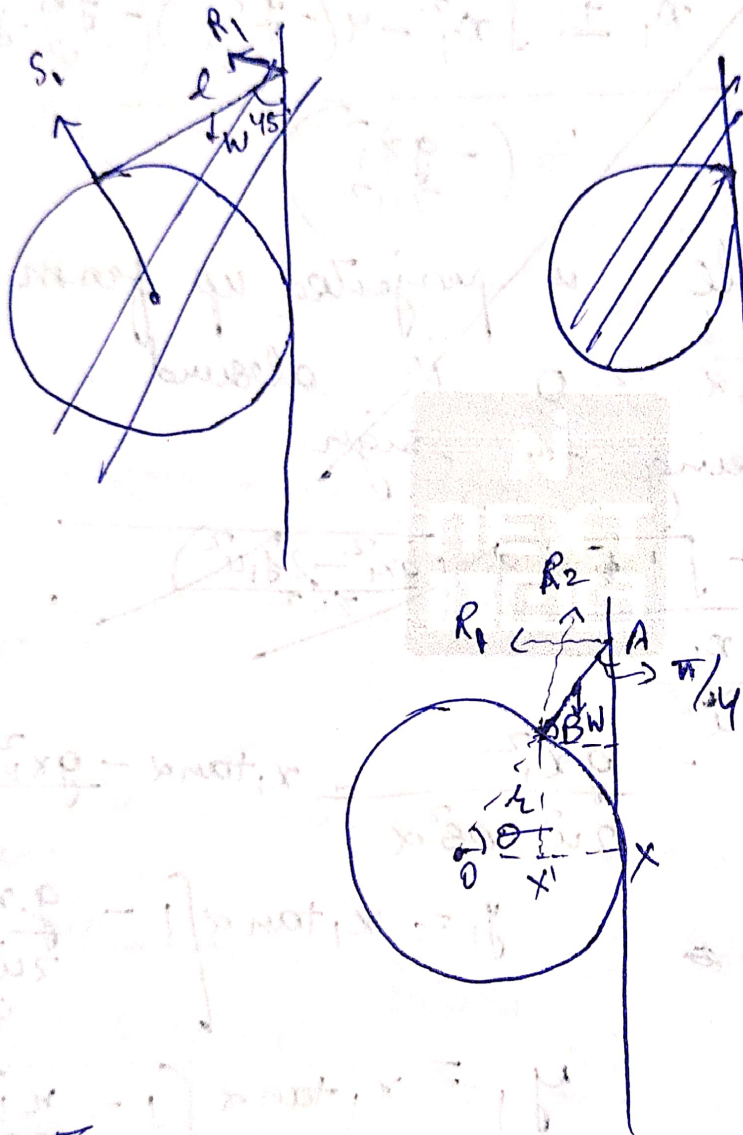
$$\Rightarrow \tan \alpha = \frac{y_1 R}{x_1 R - x_1^2}$$

Q.5 (e)

A cylinder of radius ' r ' whose axis is fixed horizontally, touches a vertical wall along a generating line. A flat beam of length ' l ' and weight W rests with its extremities in contact with the wall and the cylinder, making an angle 45° with the vertical. Prove that the reaction of the cylinder is $\frac{W\sqrt{5}}{2}$ and the pressure on the wall is $W/2$. Also prove that the

ratio of radius of the cylinder to the length of the beam is $(5 + \sqrt{5}) : 4\sqrt{2}$.

(10)



Force balance

$$R_1 = R_2 \cos \theta \quad (\text{in horizontal})$$

$$W = R_2 \sin \theta \quad (\text{in vertical})$$

⊗ Moment balance about B

$$\frac{Wl}{2} \cos 45 = R_1 l \sin 45$$

$$R_1 = W/2$$

$$\text{Also } \frac{R_1}{W} = \tan \theta \Rightarrow \tan \theta = 2$$

$$\sin \theta = \frac{2}{\sqrt{5}}$$

$$\cos \theta = \frac{1}{\sqrt{5}}$$

$$\Rightarrow R_2 = \frac{R_1}{\cos \theta} = \frac{\sqrt{5}W}{2}$$

Pressure is $W/2$ (R_1)
Reaction is $R_2 = \frac{\sqrt{5}W}{2}$

$$OX = R = OX' + X'X = R \cos \theta + AB \cos \theta$$

$$R = \frac{R}{\sqrt{5}} + \frac{l}{\sqrt{2}}$$

$$R \left(1 - \frac{1}{\sqrt{5}}\right) = \frac{l}{\sqrt{2}}$$

$$\frac{R}{l} = \frac{\sqrt{5}}{\sqrt{2}(\sqrt{5}-1)}$$

$$= \frac{\sqrt{5}(\sqrt{5}+1)}{4\sqrt{2}} = \frac{5+\sqrt{5}}{4\sqrt{2}}$$

Q.6 (a) Reduce $\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R$, where P, Q, R are functions of x , to normal form. Hence

solve $\frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 1)y = 3e^x \sin 2x$. (20)

normal form is

$$\frac{d^2 v}{dx^2} + Sv = S.$$

let $y = uv$ be the solution where

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = v \frac{du}{dx} + u \frac{dv}{dx} \quad \text{--- (1)}$$

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(v \frac{du}{dx} + u \frac{dv}{dx} \right) \\ &= 2 \frac{dv}{dx} \frac{du}{dx} + v \frac{d^2 u}{dx^2} + u \frac{d^2 v}{dx^2} \quad \text{--- (2)} \end{aligned}$$

Substituting (1), (2) in given equation

$$\begin{aligned} &\left(v \frac{d^2 u}{dx^2} + u \frac{d^2 v}{dx^2} \right) + 2 \frac{dv}{dx} \frac{du}{dx} + P \left[v \frac{du}{dx} + u \frac{dv}{dx} \right] \\ &+ Quv = R \end{aligned}$$

Dividing by u ,

$$\begin{aligned} &\frac{d^2 v}{dx^2} + v \left[\frac{P}{u} \frac{du}{dx} + \frac{d^2 u}{dx^2} \frac{1}{u} \right] \\ &+ \frac{2}{u} \frac{dv}{dx} \frac{du}{dx} + P \frac{dv}{dx} = \frac{R}{u} \quad \text{--- (3)} \end{aligned}$$

$$\frac{d^2u}{dx^2} + \left[\frac{p}{u} \frac{du}{dx} + 0 + \frac{1}{u} \frac{d^2u}{dx^2} \right] + \frac{d}{dx} \left[p + \frac{2}{u} \frac{du}{dx} \right] = \frac{R}{u} \quad (3)$$

Choosing u such that

$$p + \frac{2}{u} \frac{du}{dx} = 0$$

$$\text{i.e. } -p dx = \frac{2 du}{u}$$

$$\text{i.e. } -\int p dx = 2 \log u \Rightarrow u = e^{\frac{\int p dx}{2}}$$

$$\frac{du}{dx} = -\frac{pu}{2}$$

$$\begin{aligned} \frac{d^2u}{dx^2} &= \frac{1}{2} \left[u \frac{dp}{dx} + p \frac{du}{dx} \right] \\ &= \frac{1}{2} \left[e^{-\frac{\int p dx}{2}} \frac{dp}{dx} - \frac{p^2 u}{2} \right] \\ &= \frac{1}{2} \left[\frac{p^2 u}{2} - e^{-\frac{\int p dx}{2}} \frac{dp}{dx} \right] \end{aligned}$$

Substituting in (3)

$$\frac{d^2u}{dx^2} + \left[-\frac{p^2}{2} + 0 + \frac{e^{\frac{\int p dx}{2}}}{2} \left[\frac{p^2 u}{2} - e^{-\frac{\int p dx}{2}} \frac{dp}{dx} \right] \right] = \frac{R}{u}$$

$$\frac{d^2u}{dx^2} + I u = \frac{R}{u} = S. \quad \text{is normal form.}$$

$$I = -\frac{p^2}{2} + Q + \frac{p^2}{4} - \frac{dp}{2dx}$$

$$= Q - \frac{1}{2} \frac{dp}{dx} - \frac{p^2}{4}$$

$$\frac{d^2y}{dx^2} + Py = S$$

$$\text{where } I = Q - \frac{1}{2} \frac{dp}{dx} - \frac{p^2}{4}$$

$$S = R/u; \quad u = e^{-\int \frac{p}{2} dx}$$

To solve

$$\frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2} \sin 2x$$

$$\text{Here } p = -4x$$

$$Q = (4x^2 - 1)$$

$$R = -3e^{x^2} \sin 2x$$

$$u = e^{-\int \frac{p}{2} dx} = e^{+2x^2}$$

$$S = \frac{R}{u} = -3e^{-x^2} \sin 2x$$

$$I = Q - \frac{1}{2} \frac{dp}{dx} - \frac{p^2}{4}$$

$$= (4x^2 - 1) - \frac{1}{2}(-4) - 4x^2 = 1$$

$$\frac{d^2 v}{dx^2} + v = -3 \sin 2x$$

$$(D^2 + 1)v = -3 \sin 2x$$

Auxiliary equation is $m^2 + 1 = 0$

$$m = \pm i$$

$$\Rightarrow v_c = c_1 e^{ix} + c_2 e^{-ix}$$

$$= c_1 (\cos x + i \sin x) + c_2 (\cos x - i \sin x)$$

$$= \cos x (c_1 + c_2) + \sin x (ic_1 - ic_2)$$

$$v_c = \boxed{A \cos x + B \sin x}$$

$A = c_1 + c_2$; $B = ic_1 - ic_2$ are arbitrary constants.

$$v_p = \frac{1}{D^2 + 1} (-3 \sin 2x) = \frac{-3 \sin 2x}{-4 + 1} = \sin 2x$$

$$\Rightarrow v = v_c + v_p = A \cos x + B \sin x + \sin 2x$$

$$\Rightarrow y = uv = e^{x^2} (A \cos x + B \sin x + \sin 2x)$$

Q.6 (b)

A particle of mass 5 units moves in a straight line towards a centre of force and the force varies inversely as the cube of distance. Starting from rest at the point A distant 20 units from centre of force O, it reaches a point B distant b from O. Find the time in reaching from A to B and the velocity at B. When will the particle reach at the centre? (15)

Given

mass = 5 units

$$\text{Force} = P \propto \frac{1}{x^3}$$

$$\Rightarrow P = -\frac{c}{x^3}$$

x is distance of particle from origin

$$\left. \frac{dx}{dt} \right|_{x=A} = 0 \quad x_A = 20 ; \quad x_B = b$$

To find $t(x_A \rightarrow x_B)$ and v_B

We know

$$\frac{d^2x}{dt^2} + u = \frac{P}{m^2 u^2}$$

$$\text{Force} = m \frac{d^2x}{dt^2} = -\frac{c}{x^3}$$

where m is mass of particle

Now multiplying both sides by

$$2 \frac{dx}{dt}$$

$$m \left(2 \frac{dx}{dt} \frac{d^2x}{dt^2} \right) = -\frac{c}{x^3} \left(2 \frac{dx}{dt} \right)$$

Integrating

$$m \left(\frac{dx}{dt} \right)^2 = \frac{-2C}{-2x^2} + C_1$$

$$\text{at } x = 20; \quad \frac{dx}{dt} = 0$$

$$\Rightarrow C_1 = \frac{-C}{20^2} = \frac{-C}{400}$$

$$\Rightarrow m \left(\frac{dx}{dt} \right)^2 = -C \left(\frac{1}{400} - \frac{1}{x^2} \right)$$

$$\frac{dx}{dt} = -\sqrt{\frac{C}{m} \left(\frac{1}{x^2} - \frac{1}{400} \right)}$$

(taking -ve sign as
x decreases with t)

$$\frac{dx}{\sqrt{\frac{1}{x^2} - \frac{1}{400}}} = -\sqrt{\frac{C}{m}} dt$$

$$\frac{20x}{\sqrt{20^2 - x^2}} dx = -\sqrt{\frac{C}{m}} dt$$

$$\text{let } 20^2 - x^2 = u$$

$$\Rightarrow -2x dx = du$$

$$\int_{x=10}^{x=b} \frac{10 du}{\sqrt{u}} = \int_{t=0}^t -\sqrt{\frac{C}{m}} dt$$

$$2 \left[\sqrt{u} \right]_{x=10}^{x=b} = -t \sqrt{\frac{C}{m}}$$

$$2 \left[\sqrt{20^2 - x^2} \right]_{x=20}^b = t \sqrt{\frac{c}{m}}$$

$$2 \left(\sqrt{400 - b^2} - \cancel{\sqrt{300}} \right) = t \sqrt{\frac{c}{5}}$$

$$\Rightarrow t_B = \frac{\sqrt{20} \left(\sqrt{400 - b^2} - \cancel{\sqrt{300}} \right)}{\sqrt{c}}$$

time to reach 0

i.e. $x = 0$

$$t_0 = \frac{20\sqrt{20}}{\sqrt{c}}$$

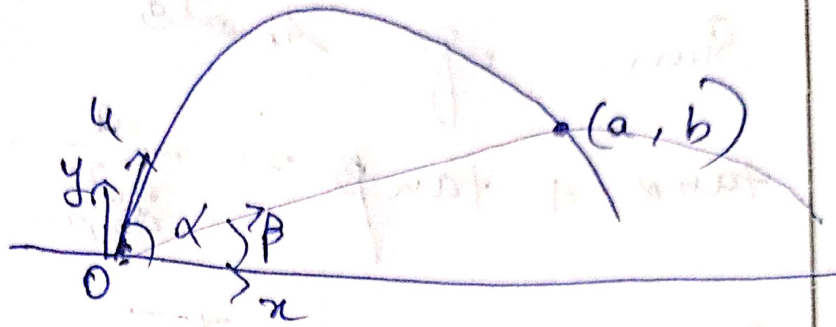
Velocity at B

$$\left(\frac{dx}{dt} \right)_B = - \sqrt{\frac{c}{5} \left(\frac{1}{b^2} - \frac{1}{400} \right)}$$

If α and β are the two possible directions to hit a given point (a, b) , then show that

$$\tan(\alpha + \beta) = -\frac{a}{b}$$

(15)



let projectile fired from O at initial velocity u at angle θ to Ox (horizontal)

We know trajectory equation

$$y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$$

at $x = a, y = b$

$$b = a \tan \theta - \frac{ga^2}{2u^2} (1 + \tan^2 \theta)$$

$$\therefore \sec^2 \theta = 1 + \tan^2 \theta$$

$$\tan^2 \theta \left(\frac{ga^2}{2u^2} \right) + \tan \theta (-a) + \frac{ga^2}{2u^2} + b = 0$$

Quadratic in $\tan \theta$

This means there are two roots $\tan \theta_1, \tan \theta_2$ i.e. 2 directions

Let these roots be α, β

Sum of roots

$$\tan \alpha + \tan \beta = \frac{a}{\frac{ga^2}{2u^2}}$$

Product of roots

$$\tan \alpha \tan \beta = \frac{\frac{ga^2}{2u^2} + b}{\frac{ga^2}{2u^2}}$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$= \frac{a/ga^2/2u^2}{1 - \frac{ga^2}{2u^2} + b}$$

$$= \frac{a}{-b} = -\frac{a}{b}$$

$$\Rightarrow \tan(\alpha + \beta) = -\frac{a}{b}$$