

NEXT IAS

(To be filled by candidate)

Name of Candidate : ANNU KOUSIK T-3/22

Roll No. : OAC24MTTA003 Date of Examination : 7/07/2024

Mobile Number : _____ E-mail : _____

Exam Centre : Old Rajinder Nagar ☒ Online ☐

CSE (MAINS) TEST SERIES-2024

Test-03

MATHEMATICS
OPTIONAL

Dated : 07-07-2024

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are **EIGHT** questions divided in **Two** sections and printed in **ENGLISH**.

Candidate has to attempt **FIVE** questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

DELHI CENTRE:

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	20			
2. (c)	21	15			
3. (a)	24	20			
3. (b)	28	15			
3. (c)	31	15			
4. (a)	34	15			
4. (b)	37	20			
4. (c)	41	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	20			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	81	20			
Grand Total					

Remarks

Observations:

SECTION-A

Q.1 (a) Find the singular solution of the differential equation $9p^2(2-y)^2 - 4(3-y) = 0$. (10)

Given eqn of DE : $9p^2(2-y)^2 - 4(3-y) = 0$

$$\Rightarrow 9p^2 = \frac{4(3-y)}{(2-y)^2} \Rightarrow p^2 = \frac{4(3-y)}{9(2-y)^2}$$

$$\Rightarrow p = \frac{\sqrt{4(3-y)}}{3(2-y)} = \frac{dy}{dx}$$

$$\Rightarrow dx = \frac{6-3y}{\sqrt{12-4y}} dy \text{ Integrating,}$$

$$\int dx = \int \frac{6-3y}{\sqrt{12-4y}} dy \text{ let } y = 3 \sin^2 \theta$$

$$\Rightarrow dy = 6 \sin \theta \cos \theta d\theta$$

$$\Rightarrow x = \int \frac{6-9 \sin^2 \theta}{\sqrt{12} \cos \theta} (6 \sin \theta \cos \theta) d\theta$$

$$= \frac{6}{\sqrt{12}} \int 6 \sin \theta - 9 \sin^3 \theta d\theta$$

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- Q.1 (b) Using Laplace Transform, solve the IVP $ty'' + 2ty' + 2y = 2$; $y(0) = 1$ and $y'(0)$ is arbitrary. Does this problem has a unique solution. (10)

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Applying Laplace transform on both sides,

$$L(t y'' + 2t y' + 2y) = L(2) \quad \text{--- (1)}$$

$$\text{We know that } L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} \bar{f}(s)$$

$$\cancel{L(y'') = s^2 \bar{y}(s) - s y(0) - y'(0) = s^2 \bar{y}(s) - s - k \quad (\because y'(0) \text{ is arbitrary})}$$

$$\Rightarrow \cancel{L(t y'') = -\frac{d}{ds}(s^2 \bar{y}(s) - s - k) = -(2s \bar{y}(s) - s^2 \bar{y}'(s) + 1)}$$

$$\text{If } L(y') = s \bar{y}(s) - y(0) = s \bar{y}(s) - 1 + s \bar{y}'(s)$$

$$\Rightarrow \cancel{L(2t y') = 2 \left(\frac{d}{ds}(s \bar{y}(s) - 1) \right) = -2(\bar{y}(s) + s \bar{y}'(s))}$$

$$L(y'') = s^2 \bar{y}(s) - s y(0) - y'(0) = s^2 \bar{y}(s) - s - k$$

$$\Rightarrow L(t y'') = -\frac{d}{ds}(s^2 \bar{y}(s) - s - k) = -2s \bar{y}(s) - s^2 \bar{y}'(s) + 1$$

$$\text{If } L(y') = s \bar{y}(s) - y(0) = s \bar{y}(s) - 1$$

$$\Rightarrow L(2t y') = -2 \frac{d}{ds}(s \bar{y}(s) - 1) = -2(\bar{y}(s) + s \bar{y}'(s)) = -2 \bar{y}(s) - 2s \bar{y}'(s)$$

Substituting the values in (1),

$$(-2s \bar{y}(s) - s^2 \bar{y}'(s) + 1) - 2 \bar{y}(s) - 2s \bar{y}'(s) + 2 \bar{y}(s) = \frac{2}{s}$$

$$\Rightarrow \bar{y}(s)(-2s) + \bar{y}'(s)(-s^2 - 2s) + 1 = \frac{2}{s}$$

$$\Rightarrow \bar{y}'(s)(-s^2 - 2s) + \bar{y}(s)(-2s) + 1 = \frac{2}{s}$$

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$$\Rightarrow \bar{y}'(s) + \bar{y}(s) \left(\frac{2s}{s^2+2s} \right) - \frac{1}{s^2+2s} = \frac{2}{s}.$$

$$\Rightarrow \bar{y}'(s) + \bar{y}(s) \left(\frac{2}{s+2} \right) = \frac{2}{s} + \frac{1}{s^2+2s}.$$

$$IF = e^{\int \frac{2}{s+2} ds} = e^{\ln(s+2)^2} = (s+2)^2.$$

$$\bar{y}(IF) = \int \left(\frac{2}{s} + \frac{1}{s^2+2s} \right) \times IF + C$$

$$\Rightarrow \bar{y}(s+2)^2 = \int \frac{2s+5}{s^2+2s} \times (s+2)^2 ds + C$$

$$\Rightarrow \bar{y}(s+2)^2 = \int \left(2s+9+\frac{10}{s} \right) + C = s^2+9s+10\ln s + C.$$

$$\Rightarrow \bar{y}(s) = \frac{s^2}{(s+2)^2} + \frac{9s}{(s+2)^2} + 10 \frac{\ln s}{(s+2)^2} + \frac{C}{(s+2)^2}.$$

Q.1 (c)

Evaluate $\int_{0,0}^{2,1} (10x^4 - 2xy^3)dx - 3x^2y^2 \frac{dy}{dx}$ along the path $x^4 - 6xy^3 = 4y^2$. (10)

~~Solution type the question has typed directly~~

$$\int (10x^4 - 2xy^3)dx - 3x^2y^2 \frac{dy}{dx}$$

$$= \int (10x^4 - 2xy^3 - 3x^2y^2) dx$$

$$= \int (10x^4 - 2xy^3 - 3x^2y^2) dx$$

~~The function can be taken as:~~

~~$\int f \cdot dr$ where $f = (10x^4 - 2xy^3 - 3x^2y^2)i + 0j + 0k$~~

The function can be taken as:

$$\int_{0,0}^{2,1} (10x^4 - 2xy^3)dx - (3x^2y^2)dy = \int f \cdot dr$$

where $f = (10x^4 - 2xy^3)i + (-3x^2y^2)j + 0k$.

Calculating $\nabla \times f$:

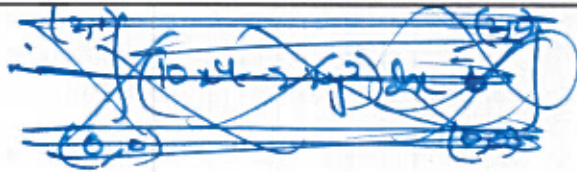
$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 10x^4 - 2xy^3 & -3x^2y^2 & 0 \end{vmatrix} = i(0) - j(0)$$

$$+ k \left(\frac{\partial}{\partial x}(-3x^2y^2) - \frac{\partial}{\partial y}(10x^4 - 2xy^3) \right)$$

$$= k(-6xy^2 - (-6xy^2)) = 0 \vec{k} = \underline{0}$$

$$\therefore (\nabla \times f) = 0$$

Hence the integral is independent of path & conservative

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$$\therefore \int_{(0,0)}^{(2,3)} (10x^4 - 2xy^3) dx - 3x^2y^2 dy = \int_{(0,0)}^{(2,0)} f \cdot dr + \int_{(2,0)}^{(2,3)} f \cdot dr$$

$$\int_{(0,0)}^{(2,0)} f \cdot dr = \int_{(0,0)}^{(2,0)} 10x^4 dx \quad (\because y = dy = 0)$$

$$= 10 \times \left(\frac{x^5}{5} \right)_0^2 = 64$$

$$\int_{(2,0)}^{(2,3)} f \cdot dr = \int_{(2,0)}^{(2,3)} -12y^2 dy \quad (\because x = 2, dx = 0)$$

$$= -12 \times \left(\frac{y^3}{3} \right)_0^3 = -4$$

$$\therefore \int_{(0,0)}^{(2,3)} (10x^4 - 2xy^3) dx - 3x^2y^2 dy$$

$$= 64 - 4 = \underline{\underline{60 \text{ units}}}$$

Q.1 (d)

If mass density at any point of a cord varies as the radius of curvature of the curve in which it hangs freely under gravity, then prove that this curve is the catenary of uniform strength. (10)

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Q.1 (e)

A train of mass M kg is ascending a smooth incline of 1 in n and velocity is v m/s, its acceleration is f m/s². Prove that the effective power of the engine is $Mv(nf + g)/n$ watts.
(10)

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- Q.2 (a) Show that the equation $(4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$ represents family of hyperbolas having as asymptotes the lines $x + y = 0$, $2x + y + 1 = 0$. (15)

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The eqn is of the form: $Mdx + Ndy = 0$.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (4x + 3y + 1) = 3;$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (3x + 2y + 1) = 3;$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

↳ The DE is exact.

The solution is given by $v + u = c$ where.

$$v = \int M dx \text{ and } u = \int \left(N - \frac{\partial v}{\partial y} \right) dy.$$

$$v = \int M dx = \int (4x + 3y + 1) dx = \underline{2x^2 + 3xy + x}$$

$$u = \int \left(N - \frac{\partial v}{\partial y} \right) dy = \int (3x + 2y + 1 - 3x) dy = \underline{y^2 + y}$$

$$\Rightarrow \text{the solution is: } \underline{2x^2 + y^2 + 3xy + x + y = c} \quad \text{--- (1)}$$

$$\Rightarrow (2x + y + 1)(x + y) = c.$$

$\Rightarrow$ The curve has asymptotes as

the lines: $(2x + y + 1) = 0$ and

$$(x + y) = 0.$$

Comparing ① with

$$cz^2 + ax^2 + by^2 + 2hxy + 2fyz + 2gzx + 4x + vy + wz = d,$$

$$a=2, b=1, h=\frac{3}{2}, f, g=0, c=0.$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \quad \& \quad h^2 - ab > 0.$$

$\Rightarrow$ The curve clearly represents the
eqn of family of hyperbolas with
 $2x+y+1=0$ & $x+y=0$ as the
asymptotes.

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Q.2 (b)

Find the values of a , b and c such that the maximum value of directional derivative of $f = axy^2 + byz + cx^2z^2$ at $(1, -1, 1)$ is in the direction parallel to y axis and has magnitude 6.

(20)

~~Given that f satisfies the point $(1, -1, 1)$~~

~~$$\Rightarrow axy^2 + byz + cx^2z^2 \Big|_{(1, -1, 1)} = 0$$~~

~~$$\Rightarrow a - b + c = 0 \quad \text{--- (1)}$$~~

Finding the directional derivative via ∇f :

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$= i \frac{\partial}{\partial x} (axy^2 + byz + cx^2z^2) + j \frac{\partial}{\partial y} (axy^2 + byz + cx^2z^2) + k \frac{\partial}{\partial z} (axy^2 + byz + cx^2z^2)$$

$$= i (ay^2 + 2cxz^2) + j (2axy + bz) + k (by + 2cx^2z)$$

$$\nabla f \Big|_{(1, -1, 1)} = i (a + 2c) + j (-2a + b) + k (-b + 2c) \quad \text{--- (2)}$$

Now ∇f has maximum value of directional derivative $\parallel$ to y axis.

$\Rightarrow \nabla f \Big|_{(1, -1, 1)}$ should be $\parallel$ to y -axis.

$$\Rightarrow (a + 2c) = 0 \text{ and } (-b + 2c) = 0 \quad \text{--- (3)}$$

(Components of i and k in (2) should be zero)

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From ③, $a+2c=0$ and $-b+2c=0$.

Let $c=k \Rightarrow a=-2k$ and $b=2k$.

$$\therefore \nabla f|_{(1,-1,1)} = (-2a+b)\hat{j} = (6k)\hat{j} \quad \text{--- ④}$$

Magnitude of directional derivative along any vector $\vec{P}$ is given by

$$\text{dir-derivative} = \nabla f \cdot \left[\frac{\vec{P}}{|\vec{P}|} \right]$$

$\therefore$ Directional derivative of ∇f along $\vec{j}$ is 6.

$$\Rightarrow \left| \nabla f \right| \cdot \left[\frac{\vec{j}}{|\vec{j}|} \right] = 6 \Rightarrow \nabla f \cdot \vec{j} = 6 \quad (\because |\vec{j}|=1)$$

$$\Rightarrow (6k\vec{j}) \cdot \vec{j} = 6 \Rightarrow 6k = 6 \Rightarrow k = \pm 1$$

$$\Rightarrow \underline{C = k = \pm 1}$$

$$a = -2k = \mp 2$$

$$b = 2k = \pm 2$$

$$\therefore (a, b, c) = (\mp 2, \pm 2, \pm 1)$$

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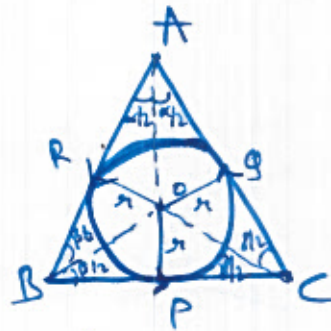
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Q.2 (c)

Three forces P , Q and R act along the sides BC , CA and AB of $\triangle ABC$ in order to keep the system in equilibrium. If the resultant force touches the inscribed circle, then prove that

$$\frac{1 + \cos \alpha}{P} + \frac{1 + \cos \beta}{Q} + \frac{1 + \cos \gamma}{R} = 0. \quad (15)$$

Given that the
resultant force ($\vec{T}$)
touches the incircle



$$\Rightarrow \vec{T} \times \vec{r} = \vec{P} \times (\vec{r}) + \vec{Q} \times (\vec{r}) + \vec{R} \times (\vec{r}) = 0$$

(Moment of individual forces & the
resultant force)

($\because$ the system is in equilibrium).

$$\text{In the } \triangle OBC, \angle BOC = \left(\pi - \frac{\beta + \gamma}{2} \right)$$

$$\Rightarrow \angle BOC = \left(\pi - \left(\frac{\pi - \alpha}{2} \right) \right) = \left(\frac{\pi}{2} + \frac{\alpha}{2} \right)$$

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Q.3 (a) Find singular solution of the differential equation $y^2 \left(\frac{dy}{dx} \right)^2 - 2xy \left(\frac{dy}{dx} \right)$

$\tan^2 \beta + y^2 \sec^2 \beta - x^2 \tan^2 \beta = 0$ directly and from its complete primitive. Determine tac locus. Show that the envelope of family of curves, which is represented by the given equation, is $y = \pm x \tan \beta$. (20)



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Q.3 (b)

A person is drawing water from a well with a light bucket which leaks uniformly. The bucket weighs 50 kg when it is full. When it arrives at the top, half of the water remains inside. If the depth of the water level in the well from the top is 30m, then find the work done in raising the bucket to the top from the water level. (15)

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Q.3 (c)

Prove that the central orbit is a plane curve.

(15)

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Q.4 (a)

Given that $y_1 = x^2$ is a solution of the differential equation, $x^3 \frac{d^2 y}{dx^2} - (x^2 + 3x) \frac{dy}{dx} + 6y = 0$, find the other linearly independent solution of the above differential equation and write down the general solution of the differential equation. (15)

$$\text{Given DE: } \frac{d^2 y}{dx^2} + \left(\frac{-x^2 - 3x}{x^3} \right) \frac{dy}{dx} + \frac{6y}{x^3} = 0$$

$$\Rightarrow \frac{d^2 y}{dx^2} + \left(-\frac{1}{x} - \frac{3}{x^2} \right) \frac{dy}{dx} + \frac{6y}{x^3} = 0$$

$$\text{Comparing with } \frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Q y = R,$$

$$P = -\frac{1}{x} - \frac{3}{x^2}, Q = \frac{6}{x^3}, R = 0.$$

Given that $u = x^2$ is a solution.

Then the other solution is given by the DE:

$$\frac{d^2 v}{dx^2} + \left(P + \frac{2}{u} \frac{du}{dx} \right) \frac{dv}{dx} = \frac{R}{u}$$

$$\Rightarrow \frac{d^2 v}{dx^2} + \left(-\frac{1}{x} - \frac{3}{x^2} + \frac{2}{x^2} \cdot 2x \right) \frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2 v}{dx^2} + \left(\frac{3}{x} - \frac{3}{x^2} \right) \frac{dv}{dx} = 0. \quad \text{Let } \frac{dv}{dx} = p. \text{ Then } \frac{d^2 v}{dx^2} = \frac{dp}{dx}.$$

$$\Rightarrow \frac{dp}{dx} + \left(\frac{3}{x} - \frac{3}{x^2} \right) p = 0$$

$$\Rightarrow \frac{dp}{p} = \left(\frac{3}{x^2} - \frac{3}{x} \right) dx$$

$$\Rightarrow \frac{dp}{p} = dx \left(\frac{3}{x^2} - \frac{3}{x} \right). \text{ Integrating,}$$

$$\Rightarrow \int \frac{dp}{p} = 3 \int \frac{1}{x^2} - \frac{1}{x} dx$$

$$\Rightarrow \ln p = 3 \left(-\frac{1}{x} - \ln x \right)$$

$$\Rightarrow \ln p = -\frac{3}{x} - \ln x^3 = -\frac{3}{x} + \ln \frac{1}{x^3}$$

$$\Rightarrow p = \frac{dv}{dx} = e^{-\frac{3}{x} + \ln(\frac{1}{x^3})} = e^{-3/x} \times \frac{1}{x^3}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{x^3} e^{-3/x} \Rightarrow dv = \frac{e^{-3/x}}{x^3} dx$$

Integrating on both sides,

$$\int dv = \int \frac{e^{-3/x}}{x^3} dx \quad \text{Let } \frac{3}{x} = t$$

$$\Rightarrow x = 3/t$$

$$\Rightarrow dx = -\frac{3}{t^2} dt.$$

$$\Rightarrow \int dv = v = \int \frac{e^{-t}}{(3/t)^3} \times -\frac{3}{t^2} dt.$$

$$\Rightarrow v = -\frac{3}{27} \int t e^{-t} dt = -\frac{1}{9} \int t e^{-t} dt.$$

$$\Rightarrow v = -\frac{1}{9} \left[t \int e^{-t} - \int \frac{d}{dt}(t) \int e^{-t} dt \right] \quad \left(\begin{array}{l} \int f g = t \int g \\ - \int t' f \end{array} \right)$$

$$\Rightarrow v = -\frac{1}{9} \left[-t e^{-t} - e^{-t} \right] = \frac{e^{-t}}{9} (t+1)$$

$$\Rightarrow \boxed{v = \frac{e^{-3/x}}{9} \left(\frac{3}{x} + 1 \right)}$$

is the other solution to the DE.

General solution of the DE is given
by $u \cdot v = C$

$$\Rightarrow x^2 \left(\frac{e^{-3/2x}}{9} \left(\frac{3}{x} + 1 \right) \right) = C = y$$

$$\Rightarrow \boxed{\frac{e^{-3/2x}}{9} (3x + x^2) = C} \text{ is the reqd general} \\ \text{soln of the DE.}$$



Q.4 (b)

$$\text{Let } F = \frac{-xi + x}{x^2 + y^2}$$

(a) Calculate $\Delta \times F$. (b) Evaluate $\oint F \cdot dr$ around any closed path and explain the results.

(20)

② Given that $F = \frac{-xi + x}{x^2 + y^2}$

~~$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-x}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}$$~~

The expression $F = \frac{-xi}{x^2+y^2} + \frac{x}{x^2+y^2}$

is INVALID as

there exist no sum of a scalar $\left(\frac{x}{x^2+y^2}\right)$ and a vector.

$$\left(\frac{-xi}{x^2+y^2}\right)$$

Hence $\nabla \times F$ does not exist as

F is ~~neither a scalar~~ not a vector function

(b) The expression $\oint F \cdot dr$
is also INVALID and
non-existent as ~~there is~~
 F is not a vector function.



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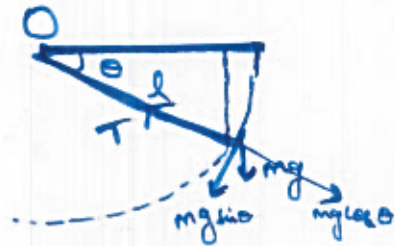


Q.4 (c)

A particle attached to a fixed peg O by a string of length l , is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with the horizontal, the resultant acceleration is $g\sqrt{1+3\sin^2\theta}$. (15)

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After descending through an angle θ , the vertical displacement of the particle from the initial position is :



~~$y = l \sin \theta$~~ $y = l \sin \theta$. By the principle of conservation of energy, loss in PE = Gain in KE

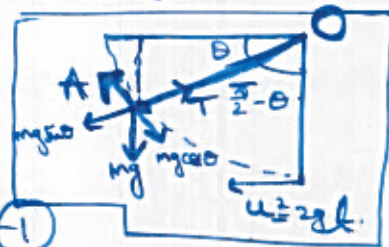
$$\Rightarrow mg l \sin \theta = \frac{1}{2} m v^2$$

$$\Rightarrow v^2 = 2gl \sin \theta. \text{ At } \theta = \pi/2, v^2 = 2gl.$$

The problem can be reassumed as the particle which is projected from the lowest point with the velocity $u^2 = 2gl$. (By symmetry) & reaches the 60° with horizontal.

The tangential component of acceleration

$$a_t = \frac{mg \cos \theta}{m} = g \cos \theta \quad (1)$$



The normal component is $a_n = \frac{T}{m} - g \sin \theta$. $\hookrightarrow (2)$

Finding T .

$$\text{We know that } T = \frac{\mu}{l} (u^2 - 2dg + 3lg \cos(\theta))$$

$$\text{At A, } T = \frac{m}{2} (2g - 2g + 3g \cos(\frac{\pi}{2} - \theta))$$

$$\Rightarrow T = m (3g \sin \theta)$$

$$\Rightarrow \frac{T}{m} = 3g \sin \theta.$$

$$\begin{aligned} \text{Now, } a_n &= \frac{T}{m} - g \sin \theta = 3g \sin \theta - g \sin \theta \\ &= 2g \sin \theta. \end{aligned}$$

$$\therefore a_t = g \cos \theta,$$

$$a_n = 2g \sin \theta$$

$$\therefore a = \sqrt{a_n^2 + a_t^2}$$

$$\Rightarrow a = \sqrt{4g^2 \sin^2 \theta + g^2 \cos^2 \theta}$$

$$\Rightarrow a = g \sqrt{\cos^2 \theta + \sin^2 \theta + 3 \sin^2 \theta}$$

$$\Rightarrow \boxed{a = g \sqrt{1 + 3 \sin^2 \theta}} \quad (\because \cos^2 \theta + \sin^2 \theta = 1)$$

Hence the required $\vec{a}$ at the point after descending through angle θ : $a = g \sqrt{1 + 3 \sin^2 \theta}$

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SECTION-B

Q.5 (a) Solve the initial value problem $(2x^2 + y)dx + (x^2y - x)dy = 0, y(1) = 2.$ (10)

Given equation: $(2x^2 + y)dx + (x^2y - x)dy = 0.$

The given equation is of the form $Mdx + Ndy = 0$
where $M = 2x^2 + y$ and $N = x^2y - x$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2x^2 + y) = 1; \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2y - x) = 2xy - 1$$

The given DE is not exact as $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$

We can see that $k = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$

$$\Rightarrow k = \frac{1}{x(xy-1)} \times (1 - (2xy-1)) = \frac{1}{x(xy-1)} (2 - 2xy)$$

$$\Rightarrow k = \frac{2}{x(xy-1)} (1 - xy) = -\frac{2}{x}.$$

$$\text{Integrating factor, IF} = e^{\int k dx} = e^{\int -\frac{2}{x} dx} \\ = e^{-2 \ln x} = \frac{1}{x^2}$$

Multiplying the IF on both
sides of the eqn,

$$\left(2 + \frac{y}{x^2}\right)dx + \left(y - \frac{1}{x}\right)dy = 0 \text{ which is an exact DE.}$$

The solution is given by $V + U = C.$

$$V = \int M dx = \int \left(2 + \frac{y}{x^2}\right) dx = 2x - \frac{y}{x} + C.$$

$$v = \left(u - \frac{\partial v}{\partial y} \right) dy = \left(y - \frac{1}{x} \right) - \left(-\frac{1}{x} \right) dy$$

$$\Rightarrow v = \int y dy = \frac{y^2}{2} \quad \text{--- (2)}$$

$u + v = C$ is the reqd soln

$$\Rightarrow 2x - \frac{y}{x} + \frac{y^2}{2} = C. \quad \text{But } y(1) = 2$$

$\Rightarrow$ At $x = 1$,

$$2(1) - \frac{y(1)}{1} + \frac{(y(1))^2}{2} = C$$

$$\Rightarrow 2 - \frac{2}{1} + \frac{4}{2} = C \Rightarrow C = 2$$

$\therefore$ The reqd solution is:

$$2x - \frac{y}{x} + \frac{y^2}{2} = 2$$

(or)

$$4x^2 - 2y + xy^2 - 4x = 0$$

- Q.5 (b) If the unit vector $\vec{T}$ and binormal $\vec{b}$ makes angle θ and ϕ respectively with a constant unit vector $\vec{a}$, prove that $\frac{\sin \theta}{\sin \phi} \cdot \frac{d\theta}{d\phi} = -\frac{\kappa}{\tau}$. (10)

Given that $\vec{T}$ makes θ with $\vec{a}$

$$\Rightarrow \frac{\vec{T} \cdot \vec{a}}{|\vec{T}| |\vec{a}|} = \cos \theta \Rightarrow \cos \theta = \vec{T} \cdot \vec{a} \quad \text{--- (1)}$$

Also $\vec{b}$ makes ϕ with $\vec{a}$.

$$\Rightarrow \frac{\vec{b} \cdot \vec{a}}{|\vec{b}| |\vec{a}|} = \cos \phi \Rightarrow \cos \phi = \vec{b} \cdot \vec{a} \quad \text{--- (2)}$$

Differentiating (1) on both sides w.r.t s :

$$\frac{d}{ds} (\cos \theta) = \frac{d}{ds} (\vec{T} \cdot \vec{a})$$

$$\Rightarrow -\sin \theta \frac{d\theta}{ds} = \frac{d\vec{T}}{ds} \cdot \vec{a} + \vec{T} \cdot \frac{d\vec{a}}{ds} \quad \left(\frac{d\vec{a}}{ds} = 0, \vec{a} \text{ is constant} \right)$$

$$\Rightarrow -\sin \theta \frac{d\theta}{ds} = \frac{d\vec{T}}{ds} \cdot \vec{a} = (\kappa \vec{n}) \cdot \vec{a} = \kappa (\vec{n} \cdot \vec{a})$$

(By Serret-Frenet formulae, $\frac{d\vec{T}}{ds} = \kappa \vec{N}$)

$$\Rightarrow \underline{-\sin \theta \frac{d\theta}{ds} = \kappa (\vec{n} \cdot \vec{a})} \quad \text{--- (3)}$$

Differentiating (2) on both sides w.r.t s :

$$\frac{d}{ds} (\cos \phi) = \frac{d}{ds} (\vec{b} \cdot \vec{a}) = \frac{d\vec{b}}{ds} \cdot \vec{a} + \vec{b} \cdot \frac{d\vec{a}}{ds}$$

($\because \frac{d\vec{a}}{ds} = 0$ as $\vec{a}$ is constant vector)

$$\Rightarrow \frac{d}{ds}(\cos \phi) = -\sin \phi \frac{d\phi}{ds} = \frac{d\vec{b}}{ds} \cdot \vec{a}$$

$$\Rightarrow -\sin \phi \frac{d\phi}{ds} = (-r\vec{n}) \cdot \vec{a} = (-r)(\vec{n} \cdot \vec{a})$$

(By Serret-Frenet formulae, $\frac{d\vec{b}}{ds} = -r\vec{n}$)

$$\Rightarrow -\sin \phi \frac{d\phi}{ds} = -r(\vec{n} \cdot \vec{a}) \quad \text{--- (4)}$$

Dividing (3) & (4),

$$\frac{-\sin \theta \left(\frac{d\theta}{ds} \right)}{-\sin \phi \left(\frac{d\phi}{ds} \right)} = \frac{k(\vec{n} \cdot \vec{a})}{(-r)(\vec{n} \cdot \vec{a})} = -\frac{k}{r}$$

$$\Rightarrow \boxed{\frac{\sin \theta}{\sin \phi} \frac{d\theta}{d\phi} = -\frac{k}{r}} \quad \text{Hence proved.}$$

Q.5 (c)

Prove that $\nabla \times (\phi A) = (\nabla \phi) \times A + \phi (\nabla \times A)$

(10)

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Let $A = A_1 i + A_2 j + A_3 k$ & $\phi A = \phi A_1 i + \phi A_2 j + \phi A_3 k$.

$$\nabla \times (\phi A) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix} = i \left(\frac{\partial}{\partial y} (\phi A_3) - \frac{\partial}{\partial z} (\phi A_2) \right)$$

$$- j \left(\frac{\partial}{\partial x} (\phi A_3) - \frac{\partial}{\partial z} (\phi A_1) \right) + k \left(\frac{\partial}{\partial x} (\phi A_2) - \frac{\partial}{\partial y} (\phi A_1) \right)$$

$$= i \left(\frac{\partial \phi}{\partial y} A_3 + \phi \left(\frac{\partial A_3}{\partial y} \right) - \left(\frac{\partial \phi}{\partial z} \right) A_2 - \phi \left(\frac{\partial A_2}{\partial z} \right) \right)$$

$$+ j \left(\frac{\partial \phi}{\partial z} A_1 + \phi \frac{\partial A_1}{\partial z} - \frac{\partial \phi}{\partial x} A_3 - \phi \frac{\partial A_3}{\partial x} \right)$$

$$+ k \left(\frac{\partial \phi}{\partial x} A_2 + \phi \frac{\partial A_2}{\partial x} - \frac{\partial \phi}{\partial y} A_1 - \phi \frac{\partial A_1}{\partial y} \right) \quad \text{--- (I)}$$

Calculating R.H.S:

$$\nabla \phi \times A = \left(i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right) \times (A_1 i + A_2 j + A_3 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = i \left(A_3 \frac{\partial \phi}{\partial y} - A_2 \frac{\partial \phi}{\partial z} \right) + j \left(A_1 \frac{\partial \phi}{\partial z} - A_3 \frac{\partial \phi}{\partial x} \right) \\ + k \left(A_2 \frac{\partial \phi}{\partial x} - A_1 \frac{\partial \phi}{\partial y} \right)$$

Calculating $\phi (\nabla \times A)$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = i \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + j \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \\ + k \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)$$

$$\phi(\nabla \times A) = i \left(\phi \frac{\partial A_3}{\partial y} - \phi \frac{\partial A_2}{\partial z} \right) + j \left(\phi \frac{\partial A_1}{\partial z} - \phi \frac{\partial A_3}{\partial x} \right) + k \left(\phi \frac{\partial A_2}{\partial x} - \phi \frac{\partial A_1}{\partial y} \right)$$

Now, $\nabla \phi \times A + \phi(\nabla \times A)$ is

$$= i \left(A_3 \frac{\partial \phi}{\partial y} + \phi \frac{\partial A_3}{\partial y} - A_2 \frac{\partial \phi}{\partial z} - \phi \frac{\partial A_2}{\partial z} \right) + j \left(A_1 \frac{\partial \phi}{\partial z} + \phi \frac{\partial A_1}{\partial z} - A_3 \frac{\partial \phi}{\partial x} - \phi \frac{\partial A_3}{\partial x} \right) + k \left(A_2 \frac{\partial \phi}{\partial x} + \phi \frac{\partial A_2}{\partial x} - A_1 \frac{\partial \phi}{\partial y} - \phi \frac{\partial A_1}{\partial y} \right)$$

→ (II)

Clearly we can see that (I) = (II)

Hence,

$$\boxed{\nabla \times (\phi A) = \phi(\nabla \times A) + (\nabla \phi \times A)}$$

Hence proved.

Q.5 (d)

A particle is projected in a direction making an angle α with the horizontal. It passes through the two points (x_1, y_1) and (x_2, y_2) . Prove that $\tan \alpha = \frac{y_1 R}{R x_1 - x_1^2} = \frac{x_2^2 y_1 - x_1^2 y_2}{(x_1 x_2)(x_2 - x_1)}$, where R denotes the horizontal range. (10)

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The equation of projectile in cartesian form is:-

$$y = x \tan \alpha - \frac{g x^2}{2u^2 \cos^2 \alpha} \text{ and } R = \frac{u^2 \sin 2\alpha}{g}$$

~~The particle passes through (x_1, y_1)~~ The particle passes through (x_1, y_1)

$$\Rightarrow y_1 = x_1 \tan \alpha - \frac{g x_1^2}{2u^2 \cos^2 \alpha} \quad \text{--- (1)}$$

$$\Rightarrow y_1 = x_1 \tan \alpha - \frac{x_1^2}{\frac{R}{\sin 2\alpha} \cos^2 \alpha} \quad \left(\because R = \frac{2u^2 \sin \alpha \cos \alpha}{g} \Rightarrow \frac{2u^2}{g} = \frac{R}{\sin 2\alpha} \right)$$

$$\Rightarrow y_1 = x_1 \tan \alpha - \frac{x_1^2}{R \cot \alpha} = x_1 \tan \alpha - \frac{x_1^2 \tan \alpha}{R}$$

$$\Rightarrow y_1 = \tan \alpha \left(\frac{R x_1 - x_1^2}{R} \right)$$

$$\Rightarrow \boxed{\tan \alpha = \frac{R y_1}{R x_1 - x_1^2}} \quad \text{Hence proved.}$$

Now the particle also passes through (x_2, y_2)

$$\Rightarrow y_2 = x_2 \tan \alpha - \frac{g x_2^2}{2u^2 \cos^2 \alpha} \quad \text{--- (2)}$$

$$\textcircled{1} \times y_2 \Rightarrow y_1 y_2 = x_1 y_2 \tan \alpha - \frac{g x_1^2 y_2}{2u^2 \cos^2 \alpha} \quad \text{--- (3)}$$

$$\textcircled{2} \times y_1 \Rightarrow x_2 y_1 \tan \alpha - \frac{g x_2^2 y_1}{2u^2 \cos^2 \alpha} = y_1 y_2 \quad \text{--- (4)}$$

From (3) & (4),

$$x_1 y_2 \tan \alpha - \frac{g x_1^2 y_2}{2 u^2 \cos^2 \alpha} = x_2 y_1 \tan \alpha - \frac{g x_2^2 y_1}{2 u^2 \cos^2 \alpha}$$

$$\Rightarrow \frac{g}{2 u^2 \cos^2 \alpha} (x_2^2 y_1 - x_1^2 y_2) = (x_2 y_1 - x_1 y_2) \tan \alpha$$

$$\Rightarrow \tan \alpha = \frac{x_2^2 y_1 - x_1^2 y_2}{\frac{2u^2}{g} \cos^2 \alpha (x_2 y_1 - x_1 y_2)}$$



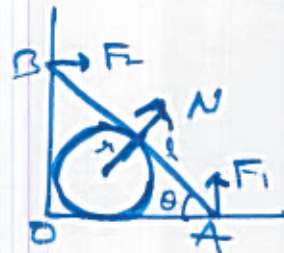
Q.5 (e)

A cylinder of radius ' r ' whose axis is fixed horizontally, touches a vertical wall along a generating line. A flat beam of length ' l ' and weight W rests with its extremities in contact with the wall and the cylinder, making an angle 45° with the vertical. Prove that the

reaction of the cylinder is $\frac{W\sqrt{5}}{2}$ and the pressure on the wall is $W/2$. Also prove that the

ratio of radius of the cylinder to the length of the beam is $(5+\sqrt{5}):4\sqrt{2}$. (10)

Taking the
moment along A



$$M_A = \cancel{N \cos \theta} (N \cos \theta) \times \frac{l}{2} \sin \theta + F_2 \times l \sin \theta = 0.$$

$$\Rightarrow \frac{-Nl}{2} \cos \theta \sin \theta + F_2 l \sin \theta = 0 \quad \text{At } \theta = \pi/4, \quad \frac{-N \cos \theta}{2}$$

$$\Rightarrow \frac{Nl}{2} \times \frac{1}{2} + \frac{F_2 l}{\sqrt{2}} = 0 \Rightarrow \frac{N}{2} + \frac{F_2}{\sqrt{2}} = 0 \quad \text{--- (1)}$$

By taking the Moments along B, we get

$$-N/2 + \frac{F_1}{\sqrt{2}} = 0 \quad \text{--- (2)} \Rightarrow F_1 = F_2$$

Equating the vertical forces to zero:

$$N \cos 45^\circ + F_1 - W = 0$$

$$\Rightarrow \frac{N}{\sqrt{2}} + F_1 = W \Rightarrow \frac{N}{2} + \frac{F_1}{\sqrt{2}} = \frac{W}{\sqrt{2}}$$

~~From (1) and (2),~~

$$\Rightarrow \frac{N}{\sqrt{2}} + F_1 = W$$

$$\Rightarrow \frac{N}{2} + \frac{F_1}{\sqrt{2}} = \frac{W}{\sqrt{2}}$$

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Q.6 (a) Reduce $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, where P, Q, R are functions of x , to normal form. Hence

solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2} \sin 2x$. (20)



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Q.6 (b)

A particle of mass 5 units moves in a straight line towards a centre of force and the force varies inversely as the cube of distance. Starting from rest at the point A distant 20 units from centre of force O , it reaches a point B distant b from O . Find the time in reaching from A to B and the velocity at B . When will the particle reach at the centre? (15)

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- Q.6 (c) If α and β are the two possible directions to hit a given point (a, b) , then show that
- $$\tan(\alpha + \beta) = -\frac{a}{b}. \quad (15)$$

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Q.7 (a)

Solve the differential equation: $(1-x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - (1+x^2)y = x$.

(15)

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Q.7 (b) Find the radii of curvature and torsion at a point of intersection of the surfaces

$$x^2 - y^2 = c^2, y = x \tanh\left(\frac{z}{c}\right).$$

(20)

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- Q.7 (c) Find the law of force for the orbit $r^2 = a^2 \cos 2\theta$ (the pole being the centre of the force).
(15)

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Q.8 (a) Solve the following initial value problem by using the Laplace transform technique:

$$\frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 3y(t) = f(t), y(0) = 1, y'(0) = 0 \text{ and } f(t) \text{ is a given function of } t. \quad (15)$$

Applying Laplace transform on both sides,

$$L(y'' - 4y' + 3y) = L(f(t)) = \bar{f}(s)$$

$$\Rightarrow (s^2 \bar{y}(s) - sy(0) - y'(0)) - 4(s\bar{y}(s) - y(0)) + 3\bar{y}(s) = \bar{f}(s)$$

$$\Rightarrow (s^2 \bar{y}(s) - s) - 4(s\bar{y}(s) - 1) + 3\bar{y}(s) = \bar{f}(s)$$

$$\Rightarrow \bar{y}(s) (s^2 - 4s + 3) - s + 4 = \bar{f}(s)$$

$$\Rightarrow \bar{y}(s) = \frac{\bar{f}(s) + s - 4}{s^2 - 4s + 3}$$

Applying Inverse Laplace transform both sides,

$$L^{-1}(\bar{y}(s)) = y(t) = L^{-1}\left(\frac{s-4}{s^2-4s+3} + \frac{\bar{f}(s)}{s^2-4s+3}\right)$$

$$L^{-1}\left(\frac{s-4}{s^2-4s+3}\right) = \frac{1}{2} L^{-1}\left((s-4)\left(\frac{1}{s-3} - \frac{1}{s-1}\right)\right)$$

$$= \frac{1}{2} L^{-1}\left(\frac{s-4}{s-3} - \frac{s-4}{s-1}\right)$$

$$= \frac{1}{2} L^{-1}\left(1 - \frac{1}{s-3} - \left(1 - \frac{3}{s-1}\right)\right) = \frac{1}{2} L^{-1}\left(\frac{3}{s-1} - \frac{1}{s-3}\right)$$

$$= \frac{1}{2} \times (3 \times e^t - e^{3t}) \quad (\because L^{-1}\left(\frac{1}{s-a}\right) = e^{at})$$

$$= \frac{3e^t - e^{3t}}{2}. \text{ Now, calculating } L^{-1}\left(\frac{f(s)}{(s-1)(s-3)}\right)$$

$$L^{-1}\left(\frac{f(s)}{s^2 - 4s + 3}\right) = \frac{1}{2} L^{-1}\left(\frac{3f(s)}{s-1} - \frac{f(s)}{s-3}\right)$$

$$\Rightarrow L^{-1}\left(\frac{f(s)}{s^2 - 4s + 3}\right) = \frac{3}{2} L^{-1}\left(\frac{f(s)}{s-1}\right) - \frac{1}{2} L^{-1}\left(\frac{f(s)}{s-3}\right)$$

Applying convolution theorem, $\mathcal{L}^{-1}(f(s)g(s)) = \int_0^t f(u)g(t-u) du$

$$\text{Here } g(s) = \frac{1}{s-1} \Rightarrow g(u) = e^u \Rightarrow g(t-u) = e^{t-u}$$

$$\Rightarrow L^{-1}\left(\frac{f(s)}{s-1}\right) = e^t \int_0^t \frac{f(u)}{e^u} du$$

$$\text{Hence } L^{-1}\left(\frac{f(s)}{s-3}\right) = e^{3t} \int_0^t \frac{f(u)}{e^{3u}} du.$$

$$\therefore y(t) = \frac{3e^t - e^{3t}}{2} + e^t \int_0^t \frac{f(u)}{e^u} du + e^{3t} \int_0^t \frac{f(u)}{e^{3u}} du$$

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Q.8 (b) Given that C is a curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x + y + z = 2$ and C is described counterclockwise. Verify Stokes Theorem for the line integral

$$\int_C (-y^3 dx + x^3 dy - z^3 dz). \quad \Rightarrow F = -y^3 \hat{i} + x^3 \hat{j} - z^3 \hat{k} \quad (15)$$

$$dr = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

By Stokes theorem,

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot \hat{n} \, ds.$$

Calculating $\iint_S (\nabla \times F) \cdot \hat{n} \, ds.$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = \hat{i} \left(\frac{\partial}{\partial y} (-z^3) - \frac{\partial}{\partial z} (x^3) \right) - \hat{j} \left(\frac{\partial}{\partial x} (-z^3) - \frac{\partial}{\partial z} (-y^3) \right) + \hat{k} \left(\frac{\partial}{\partial x} (x^3) - \frac{\partial}{\partial y} (-y^3) \right)$$

$$= \hat{k} (3x^2 + 3y^2). \text{ The normal vector is}$$

the normal to the surface of plane $x + y + z = 2$

$$\Rightarrow \hat{n} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}.$$

$$\Rightarrow (\nabla \times F) \cdot \hat{n} = \hat{k} (3x^2 + 3y^2) \cdot \left(\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}} \right) = \sqrt{3} (x^2 + y^2)$$

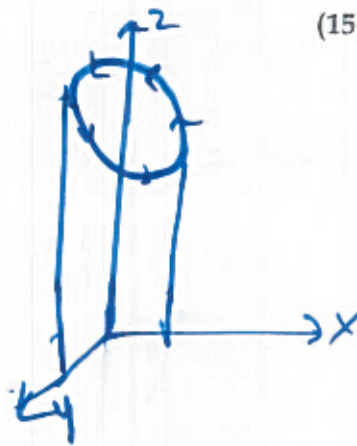
$$\iint_S (\nabla \times F) \cdot \hat{n} \, ds = \sqrt{3} \iint (x^2 + y^2) \, ds. \text{ Projecting the}$$

surface onto xy plane, $(\hat{n} \cdot \hat{k}) \, ds = dx \, dy.$

$$\Rightarrow \left[\left(\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}} \right) \cdot \hat{k} \right] ds = dx \, dy$$

$$\Rightarrow ds = \sqrt{3} \, dx \, dy.$$

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$$\begin{aligned}
 \therefore \iint (\nabla \times F) \cdot \hat{n} \, ds &= \iint \sqrt{3} \times \sqrt{3} (x^2 + y^2) \, dx \, dy \\
 &= 3 \iint x^2 + y^2 \, dx \, dy \quad \text{Converting to polar coord.} \\
 \iint \nabla \times F \cdot n \, dy &= 3 \int \int_0^{2\pi} \int_0^2 r^2(r) \, dr \, d\theta = 3 \int_0^{2\pi} \int_0^2 r^3 \, dr \, d\theta \\
 &\quad (\because dx \, dy = r \, dr \, d\theta) \\
 &= 3 \times 2\pi \times \left(\frac{r^4}{4}\right)_0^2 = 6\pi \times 4 = 24\pi
 \end{aligned}$$

$$\therefore \boxed{\iint (\nabla \times F) \cdot n \, dy = 24\pi} \quad \text{--- (I)}$$

Calculating $\oint f \cdot dr$:

$$\oint f \cdot dr = \int -y^3 \, dx + x^3 \, dy - z^3 \, dz$$

(since z^3 is an odd function, therefore, $\oint z^3 \, dz = 0$ along any closed curve).

$$\Rightarrow \oint f \cdot dr = \oint -y^3 \, dx + x^3 \, dy \quad \text{where } C \text{ is the Circle } x^2 + y^2 = 4.$$

~~Let~~ Converting to polar Co-ordinates,

$$x = 2 \cos \theta, y = 2 \sin \theta, dx = -2 \sin \theta \, d\theta$$

$$\begin{aligned}
 \therefore \oint -y^3 \, dx + x^3 \, dy &= 4 \int_0^{2\pi} -8 \sin^3(-2 \sin \theta \, d\theta) \\
 &\quad + 8 \cos^3 \theta (2 \cos \theta \, d\theta)
 \end{aligned}$$

$$= 4 \int_0^{2\pi} 16 \sin^4 \theta \, d\theta + 16 \cos^4 \theta \, d\theta$$

$$= 64 \int_0^{2\pi} \sin^4 \theta \, d\theta + \cos^4 \theta \, d\theta$$

$$= 64 \times 2 \int_0^{\frac{\pi}{2}} \sin^4 \theta d\theta \quad \left(\because \int_0^{\frac{\pi}{2}} \sin^4 \theta = \int_0^{\frac{\pi}{2}} \sin^4 \left(\frac{\pi}{2} - \theta \right) = \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta \right)$$

$$~~= 64 \times 2~~ = 128 \int_0^{\frac{\pi}{2}} \sin^4 \theta d\theta$$

$$= 128 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= 24\pi$$

$$\therefore \oint_C \mathbf{F} \cdot d\mathbf{r} = 24\pi$$

— (II)

Hence, from (I) & (II), $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iiint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dV$

Therefore Stokes theorem is verified.

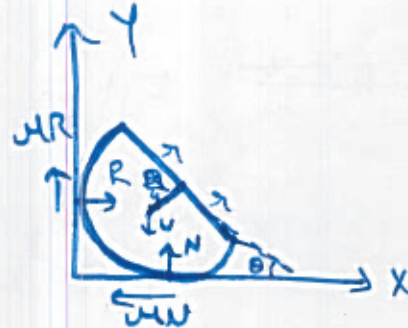
Q.8 (c)

A semi-circular disc rests in a vertical plane with its curved edge on a rough horizontal and equally rough vertical plane, the coefficient of friction being μ . Show that the greatest

angle that the bounding diameter can make with the horizontal plane is $\sin^{-1} \left(\frac{3\mu + \mu^2}{4(1 + \mu^2)} \right)$.

(20)

Let the bounding diameter make θ with the horizontal



Let R be the radius of the disc.

Then equating the vertical forces to zero,
 $\mu R + W + N = 0 \Rightarrow \underline{W = N + \mu R}$

Equating the horizontal forces to zero,
 $R - \mu N = 0 \Rightarrow \mu N = R$

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SPACE FOR ROUGH WORK

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$$T - mg \cos \theta$$

$$V^2 = u^2 - 2ag + 2ag \cos \theta$$

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \text{...} & \text{...} & \text{...} \end{vmatrix}$$

~~$$= 6mg \dots$$~~

$$\frac{s-4}{s^2-4s+3} = \frac{(s-4)}{2} \left(\frac{1}{s-3} - \frac{1}{s-1} \right)$$

$$\frac{s-1-3}{s-1} = \frac{1}{2} \left(\frac{1}{s-3} - \frac{1}{s-1} \right)$$

$$\frac{s-3-1}{s-1} = 1 - \frac{3}{s-1}$$

$$\frac{N}{2} + F_1/\sqrt{2} = W/\sqrt{2}$$

$$-\frac{N}{2} + \frac{F_1}{\sqrt{2}} = 0$$

$$\sqrt{2}F_1 = \frac{W}{2} \Rightarrow \frac{F_1}{W} = \frac{1}{2}$$