



(To be filled by candidate)

Name of Candidate : Madhukant

Roll No. : Contact No.

e-mail: Date of Examination :

Exam Centre : Old Rajinder Nagar ☐Online ☒

CSE (MAINS) TEST SERIES-2024

Test-02

MATHEMATICS OPTIONAL

Dated : 30-06-2024

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are **EIGHT** questions divided in **Two sections** and printed in **ENGLISH**.

Candidate has to attempt **FIVE** questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

DELHI CENTRE:

Old Rajinder Nagar: 27-B, Pusa Road, Metro Pillar no. 118, Near Karol Bagh Metro, New Delhi-110060 | Phone: 8081300200

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Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	20			
2. (b)	18	15			
2. (c)	21	15			
3. (a)	24	20			
3. (b)	27	15			
3. (c)	30	15			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	40	20			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	57	15			
6. (c)	61	15			
7. (a)	64	20			
7. (b)	67	15			
7. (c)	70	15			
8. (a)	74	20			
8. (b)	77	15			
8. (c)	81	15			
Grand Total					

Remarks

Observations:

SECTION-A

- Q.1** (a) Show that every group of order 15 is cyclic.
- (b) Show that the smallest subgroup V of A_4 containing $(1, 2)(3, 4), (1, 3)(2, 4)$ and $(1, 4)(2, 3)$ is isomorphic to Klein 4-group. (10)
- (c) If $\sum u_n$ converges, prove that $\sum \frac{u_n}{1 - u_n}, (u_n \neq 1)$ also converges. (10)
- (d) Give the economic interpretation of duality in linear programming. (10)
- (e) Obtain the first three terms of Laurent Series expansion of the function $f(z) = \frac{1}{e^z - 1}$ about the point $z = 0$ valid in the region $0 < |z| < 2\pi$. (10)
- Q.2** (a) Let S be the set of all real numbers except -1. Define $*$ on S by $a * b = a + b + ab$. Is $(S, *)$ a group? Find the solution of the equation $2 * x * 3 = 7$ in S . (20)
- (b) If $\langle a_n \rangle$ is a sequence of positive real numbers such that $a_n = \frac{1}{2}(a_{n-1} + a_{n-2}) \forall n \geq 3$, then show that $\langle a_n \rangle$ converges to $\frac{1}{3}(a_1 + 2a_2)$. (15)
- (c) Solve graphically the following LPP:
- $$\text{Max } z = 5x_1 - 3x_2$$
- subject to
- $$3x_1 + 2x_2 \leq 12$$
- $$-x_1 + x_2 \geq 1$$
- $$-x_1 + x_2 \leq 2$$
- $$x_1, x_2 \geq 0$$
- If the objective function is changed to $\text{Max } z = 6x_1 + 4x_2$, while the constraints remain the same, then comment on the number of solutions. Will $(4, 0)$ be also a solution? (15)
- Q.3** (a) Let M be a subgroup and N a normal subgroup of a group G . show that MN is a subgroup of G and MN/N is isomorphic to $M/(M \cap N)$. (20)
- (b) Every bounded sequence has at least one limit point. (15)
- (c) If all zeros of a polynomial lies in a half plane then show that zeros of derivative $P'(z)$ also lies in the same half plane. (15)

- Q.4** (a) Let G be a finite commutative group. Let $n \in \mathbb{Z}$ be such that n and order of G are relatively prime. Show that the function $\phi: G \rightarrow G$ defined by $\phi(a) = a^n$ for all $a \in G$, is an isomorphism of G onto G . (15)
- (b) Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify. (15)
- (c) A police department has the following minimal daily requirements for police officers during its six shift periods:?

Time of Day	Period	Minimal number required
2-6 am	1	22
6-10 am	2	55
10 am-2pm	3	88
2-6 pm	4	110
6-10 pm	5	44
10 pm-2 am	6	33

An officer must start at the beginning of a 4 hour shift and stay on duty for two consecutive shifts. Any one starting during period 6 stays on duty during period 1 of the next day. The objective of the police department is to always have on duty the minimal number required in a period but to do so with the least number of officers. Develop the corresponding linear programming problem. (20)

SECTION-B

- Q.5** (a) Let R be a non zero-commutative ring with unity. If every ideal of R is prime, prove that R is a field. (10)
- (b) Prove that if $u_n(x)$, $n = 1, 2, \dots$ are continuous in $[a, b]$ and if $\sum u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, $S(x)$ is continuous in $[a, b]$. (10)
- (c) Prove that $x_1 = 2$, $x_2 = 1$, $x_3 = 0$ is feasible solution to the following set of equations:
 $2x_1 - x_2 + 3x_3 = 3$
 $-6x_1 + 3x_2 + 7x_3 = -9$
 Is the solution basic? Justify your answer. If the solution is not basic, reduce it to a basic feasible one. (10)
- (d) Evaluate $\int_C \left[\frac{e^{2z}}{z^2(z^2 + 2z + 2)} + \log(z - 6) + \frac{1}{(x - 4)^2} \right] dz$ where C is the circle $|z| = 3$. State the theorem you use in evaluating the above theorem. (10)
- (e) Prove that a function which has no singularity in the finite part of the plane or at infinity is constant. (10)

Q.6 (a) Prove that the integral domain Z is UFD and a ED. (20)

(b) Show that $\int_0^\infty \frac{dx}{1+x^2 \sin^2 x}$ is divergent. (15)

(c) Starting with least cost method, find all the solutions to the following transportation problem:

		Warehouse				
		I	II	III	IV	Supply
Plants	A	8	6	5	3	18
	B	6	7	6	8	20
	C	10	8	4	5	18
	Demand	15	16	12	13	

(15)

Q.7 (a) Let K be a finite field. Show that the numbers of elements in K is p^n , where p is a prime, which is characteristic of K and $n \geq 1$ is an integer. Also, prove that $\frac{Z_3[X]}{(X^2+1)}$ is a field. How many elements does this field have? (20)

(b) Show that if $S(x) = \sum_{n=1}^\infty \frac{1}{n^3 + n^4 x^2}$, then its derivative $S'(x) = \sum_{n=1}^\infty \frac{1}{n^2 (1 + nx^2)^2} \forall x$. (15)

(c) Evaluate the residues of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at $z = 1, 2, 3$ and infinity and show that their sum is zero. (15)

Q.8 (a) Let V Show that $Q(\sqrt{3}, i)$ is the splitting field for $x^5 - 3x^3 + x^2 - 3$ where Q is the field of rational numbers. (20)

(b) Using contour integral method, prove that $\int_0^\infty \frac{x \sin mx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ma}$. (15)

(c) Apply Rouché's theorem to determine the number of roots of the equation $z^8 - 4z^5 + z^2 - 1 = 0$ that lie inside the circle $|z|=1$. (15)

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1a

let G be a group of order 15.

$$\text{i.e. } o(G) = 15$$

Now, to show G is cyclic, we need to show,

$\exists$ an element $a \in G$ s.t. $o(a) = 15$.

We know that, $o(a) \mid o(G)$

$$\Rightarrow o(a) = 1, 3, 5 \text{ or } 15:$$

(I) if $o(a) = 1 \Rightarrow a = e$ Ignore this case.

(II) if $o(a) = 15 \Rightarrow$ we got our element with order 15 i.e. $o(a) = o(G)$ and G will be cyclic.

(III) there exist subgroup/element of only order 3 or 5.

if we have at least one element of order 3 and 5. i.e. $o(a) = 3$ & $o(b) = 5$

$$\text{then } o(ab) = \text{lcm}(o(a), o(b)) = 15$$

$\therefore$ we got our element of $o(a)$.

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(IV) Now, let all element are of order 3.

then, $o(a) = 3 \quad \forall a \in G, a \neq e$

then Consider two Subgroup generated by a & b

$$s.t. \quad o(a) = o(b) = 3$$

then By 2nd fundamental theorem,

$$o(a(b)) = \frac{o(a) \cdot o(b)}{o(a \cap b)}$$

$$\Rightarrow o(a(b)) = 9 \quad (\because o(a \cap b) = 1)$$

which is not possible as they are distinct

as $9 \neq 15$

By, ~~for~~ we cannot have all element of order 3

$\therefore$ from above 4 Cases,

we know, there must exist at least

one element with $o(a) = o(9)$

& Hence G is cyclic.

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1clet $\sum u_n$ convergesTo prove: $\sum \frac{u_n}{1-u_n}$ ($u_n \neq 1$) also converges

let $a_n = \frac{u_n}{1-u_n}$.

Now, if $\sum u_n$ converges, thenBy nth term test $\lim_{n \rightarrow \infty} u_n = 0$

$$\begin{aligned} \text{Now consider } \lim_{n \rightarrow \infty} \frac{u_n}{a_n} &= \lim_{n \rightarrow \infty} \frac{u_n(1-u_n)}{u_n} \\ &= \lim_{n \rightarrow \infty} (1-u_n) \\ &= 1 \text{ (finite)} \end{aligned}$$

 $\therefore$ By limit Comparison Test $\sum u_n$ and $\sum a_n$ converges/diverges together

$$\therefore \sum u_n \text{ converges} \Rightarrow \underline{\underline{\sum \frac{u_n}{1-u_n} \text{ also converges}}}}$$

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के लिए1-e

Given $f(z) = \frac{1}{e^z - 1}$.

To find, Laurent Series expansion of $f(z)$
about $z=0$ in region $0 < |z| < 2\pi$

let $f(z) = \sum_{n=-\infty}^{\infty} b_n (z)^n$ be the Laurent
series of $f(z)$

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के लिए2-alet $S = \{\text{all real no except } -1\}$.and $a * b = a + b + ab$.To check $(S, *)$ is group.

① if binary operation is define.

let $a_1 = a_2$ and $b_1 = b_2$, $a_1, b_1, a_2, b_2 \in S$ then $a_1 * b_1 = a_1 + b_1 + a_1 b_1$ & $a_2 * b_2 = a_2 + b_2 + a_2 b_2$
 $= a_1 + b_1 + a_1 b_1$ clearly, $a_1 * b_1 = a_2 * b_2$

② Closure

let, $a, b \in S$.then $a * b = a + b + ab$ if $a * b = -1$ then, $a + b + ab + 1 = 0$

$$a(1+b) + 1(b+1) = 0$$

$$\Rightarrow (a+1)(b+1) = 0$$

$$\Rightarrow a = -1 \text{ or } b = -1 \text{ but, } a, b \in S$$

$$\therefore a \neq -1, \text{ and } b \neq -1$$

$$\therefore a * b \in S. \Rightarrow \underline{\text{Closure}} \checkmark$$

③ Associative.

$$(a * b) * c = (a + b + ab) * c$$

$$= a + b + ab + c + ac + bc + abc$$

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$$\begin{aligned}
 &= a + (b + c + bc) + ab + ac + abc \\
 &= a + (b * c) + a(b * c) \\
 &= a * (b * c)
 \end{aligned}$$

$\Rightarrow S$ is Associative.

(II) Identity

$$\begin{aligned}
 \because 0 \in S \quad \therefore a * 0 &= a + 0 + a(0) \\
 &= a \\
 \therefore 0 &\text{ is Identity of } a \in S.
 \end{aligned}$$

(IV) Inverse.

$$\begin{aligned}
 \text{let } a \in S, \text{ \&any } b \in S. \text{ s.t.} \\
 a * b &= a + b + ab = 0.
 \end{aligned}$$

$$b = \frac{-a}{1+a}$$

$$\because a \neq -1 \text{ and } \frac{a}{1+a} \neq 1$$

$$\therefore b = \frac{-a}{1+a} \in S$$

$$\Rightarrow \forall a \in S, \exists b \in S \text{ s.t. } \underline{a * b = 0}$$

$\therefore$ Inverse exist

$\therefore (S, *)$ is a group.

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के लिएTo find Solution of $2 * x * 3 = 7$

$$\text{we know, } 2^{-1} = -\frac{2}{1+2} = -\frac{2}{3}.$$

$$3^{-1} = -\frac{3}{1+3} = -\frac{3}{4}.$$

$$\text{Now, } 2^{-1} * 2 * x * 3 * 3^{-1} = 2^{-1} * 7 * 3^{-1}$$

$$\begin{aligned} \Rightarrow x &= 2^{-1} * 7 * 3^{-1} \\ &= \left(-\frac{2}{3} * 7\right) * -\frac{3}{4} \\ &= \left(\frac{5}{3}\right) * \left(-\frac{3}{4}\right) = -\frac{1}{3}. \end{aligned}$$

$$\therefore \boxed{x = -\frac{1}{3} \in S.}$$

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2bGiven: $\langle a_n \rangle$ is a sequence. $a_n \geq 0$.

$$a_n = \frac{1}{2} (a_{n-1} + a_{n-2}) \quad \forall n \geq 3.$$

To show $\langle a_n \rangle$ converges to $\frac{1}{3} (a_1 + 2a_2)$.

Consider, $a_3 = \frac{a_1 + a_2}{2}$.

$\therefore a_1, a_2, a_3 \geq 0$ and a_3 is mean of a_1, a_2

i.e. ~~$a_3 \leq \min$~~

let $L = \min(a_1, a_2)$ and $H = \max(a_1, a_2)$

then $L < a_3 < H$, lly, $L < a_4 < H$.

Now, let us assume $L < a_n < H$

and $L < a_{n-1} < H$

$$\therefore 2L < a_n + a_{n-1} < 2H$$

$$\& \quad L < \frac{a_n + a_{n-1}}{2} < H$$

$$\Rightarrow L < a_{n+1} < H$$

$\therefore$ By principle of Mathematical Induction,

$\langle a_n \rangle$ is bounded

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$$a_n - a_{n-1} = \frac{1}{2}(a_{n-2} - a_{n-1})$$

$$a_{n-1} - a_{n-2} = \frac{1}{2}(a_{n-3} - a_{n-2})$$

$$\Rightarrow (a_n - a_{n-1}) = \left(-\frac{1}{2}\right)^{n-2} (a_{n-3} - a_{n-2})$$

$$= \frac{(-1)^{n+1}}{2^n} (a_2 - a_1)$$

$$a_n - a_{n-1} = \frac{1}{2}(a_{n-2} - a_{n-1})$$

$$a_{n-1} - a_{n-2} = \frac{1}{2}(a_{n-3} - a_{n-2})$$

$$a_3 - a_2 = \frac{1}{2}(a_1 - a_2)$$

on adding

$$a_n - a_2 = \frac{1}{2}(a_1 - a_{n-1})$$

$$\text{as } n \rightarrow \infty \quad a_n = a_{n-1} = 1$$

$$1 - a_2 = \frac{1}{2}(a_1 - 1)$$

$$21 - 2a_2 = a_1 - 1$$

$$\Rightarrow \boxed{1 = \frac{1}{3}(a_1 + 2a_2)}$$

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$\therefore \langle a_n \rangle$ converges to $\frac{1}{3}(a_1 + 2a_2)$.

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के लिए2cGiven Maximise $Z = 5x_1 - 3x_2$

Conditions $\rightarrow 3x_1 + 2x_2 \leq 12$

$-x_1 + x_2 \geq 1$

$-x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

We will form 3 straight line on x - y plane treating
 x_1 as x & x_2 as y

① $3x_1 + 2x_2 = 12$

x	4	0
y	0	6

 $(0,0)$ lies in
the region

② $-x_1 + x_2 = 1$

x	-1	0
y	0	1

 $(0,0)$ lie outside

③ $-x_1 + x_2 = 2$

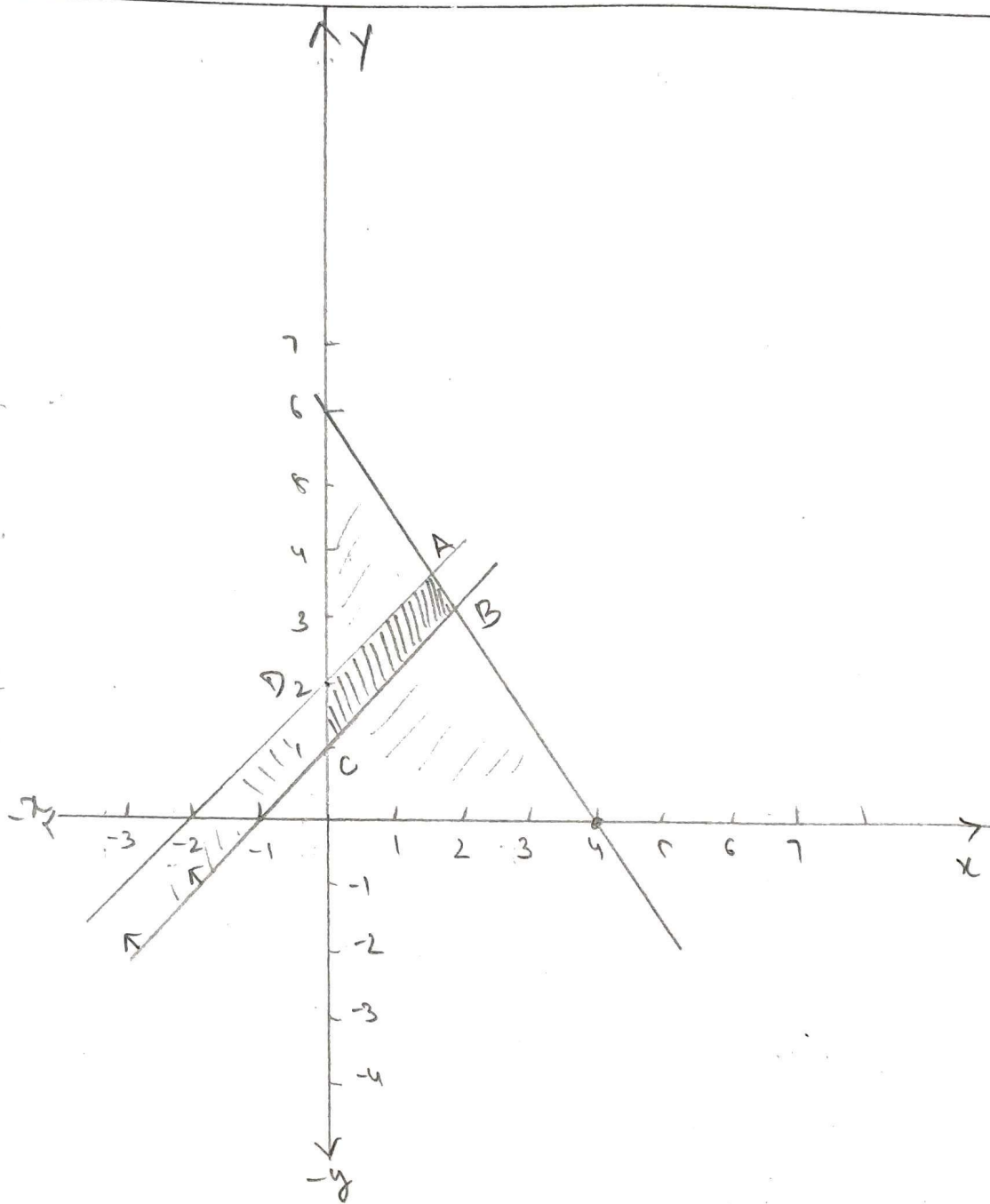
x	-2	0
y	0	2

 $(0,0)$ lie
in the region.

$x_1 \geq 0$ i.e. positive x
 and $x_2 \geq 0$ i.e. positive y

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Based on feasible region as given in shaded portion ABCD.

We know optimum solution occur at only

Corner points. , A, B, C & D.

A - Intersection of $3x + 2y = 12$ and $-x + y = 2$

$$\Rightarrow 5y = 18 \Rightarrow y = \frac{18}{5}$$

$$\Rightarrow x = \frac{8}{5} \quad A\left(\frac{8}{5}, \frac{18}{5}\right)$$

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$$5x_2 = 15 \Rightarrow x_2 = 3$$

$$\Rightarrow x_1 = 2 \quad \therefore B(2, 3)$$

& C(0, 1) & D(0, 2)

 $\therefore$ Value of optimum function Z at A $Z_A = -\frac{14}{5}$

$$Z_B = 1 \quad Z_C = -3 \quad Z_D = -6$$

 $\therefore$ Maximum value at $x_1 = 2, x_2 = 3, Z = 1$ If optm. function changes to $Z' = 6x_1 + 4x_2$

$$\text{then } Z'_A = 24$$

$$Z'_B = 24$$

$$Z'_C = 4 \quad Z'_D = 8$$

 $\therefore$ Maximum value is at A and B both. $\therefore$ problem is a Convex Set $\therefore$ Maximum value will be same on line joiningA & B. i.e. $3x_1 + 2x_2 = 12$ $\therefore$ there will be infinite solution.& $\therefore (4, 0)$ lie on $3x_1 + 2x_2 = 12$ $\therefore$ $(4, 0)$ is also a solution

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(59)

Let R be a non-zero Commutative Ring
+ R has unity

If every ideal of R is prime then

To prove: R is field.

Let us assume every ideal $I \subseteq R$ is prime
i.e. $\forall a, b \in R$ if $ab \in I$ then $a \in I$ or $b \in I$

$\therefore R$ has unity and commutative, we
need to show $\forall a \in R, a \neq 0$, a has
Inverse.

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5b

let $u_n(x)$ are continuous in $[a, b]$ Given: $\sum u_n(x)$ converges uniformly to $S(x)$.To show, $S(x)$ is continuous in $[a, b]$ let $S_n(x) = \sum_{n=1}^{\infty} u_n(x)$ be partial sum $\therefore \sum u_n(x)$ cgs uniformly $\Rightarrow S_n(x)$ cgs uniformly to $S(x)$.

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(5c)

Given equation: $2x_1 - x_2 + 3x_3 = 3$

$$-6x_1 + 3x_2 + 7x_3 = -9$$

to check if $x_1 = 2, x_2 = 1, x_3 = 0$ is a feasible solution

Substituting values. $\rightarrow$

$$2x_1 - x_2 + 3x_3 = 2(2) - 1 = 3 \text{ satisfy}$$

$$-6x_1 + 3x_2 + 7x_3 = -6(2) + 3 = -9 \text{ satisfy}$$

$\therefore$ given values are feasible solution.

$\therefore$ there are 3 equations variable and 2 equations.

and on choosing x_3 as non-basic variable

$$\text{equation will be } 2x_1 - x_2 = 3 \quad -\textcircled{1}$$

$$-6x_1 + 3x_2 = -9 \quad -\textcircled{2}$$

$$\text{eqn } \textcircled{2} = -3 \times \text{eqn } \textcircled{1} \quad \therefore \text{ its coefficient}$$

are not linearly independent

$\therefore$ solution is not basic.

To find basic solution, we choose $x_2 = 0$

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$$\begin{aligned} \text{then, } 2x_1 + 3x_3 &= 3 \\ -6x_1 + 7x_3 &= -9 \end{aligned} \quad \left. \vphantom{\begin{aligned} 2x_1 + 3x_3 &= 3 \\ -6x_1 + 7x_3 &= -9 \end{aligned}} \right\} \text{No Linearly} \\ &\quad \text{dependent Column}$$

$$\Rightarrow x_1 = \frac{3}{2} \quad \& \quad x_3 = 0$$

$\therefore x_1 = 3/2, x_2 = 0, x_3 = 0$ is basic

feasible solution.

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5d

To evaluate $\int_C \frac{e^{2z}}{z^2(z^2+2z+2)} + \log(z-6) + \frac{1}{(z-4)^2} dz$

where C is circle $|z|=3$.

$$\text{let } f_1(z) = \frac{e^{2z}}{z^2(z^2+2z+2)} \quad f_2(z) = \log(z-6)$$

$$f_3(z) = \frac{1}{(z-4)^2}$$

Consider $I_1 = \int_C f_1(z) dz$

$f_1(z)$ is analytic except at $z=0, -1+i, -1-i$

all three lie inside C .

$\therefore$ By Residue theorem - $\int_C f_1(z) dz = 2\pi i \left(\sum_{z_0} \text{Residue of } f_1 \right)$

where z_0 are singularity

$$\text{Residue at } z_0=0 = \lim_{z \rightarrow 0} \frac{d}{dz} \left(\frac{e^{2z}}{z^2+2z+2} \right)$$

$$= \lim_{z \rightarrow 0} e^{2z} \left(\frac{2}{z^2+2z+2} - \frac{(2z+2)}{(z^2+2z+2)^2} \right)$$

$$= \frac{1}{2}$$

$$\text{Residue at } z_0 = -1+i = \lim_{z \rightarrow -1+i} \frac{e^{2z}}{z(z+1+i)} = \frac{e^{-2} e^{2i}}{(-1+i)^2 (2i)}$$

$$\text{Residue at } z_0 = -1-i = \lim_{z \rightarrow -1-i} \frac{e^{2z}}{z(z+1-i)} = \frac{e^{-2} e^{-2i}}{(-1-i)^2 (-2i)}$$

$$\therefore \int_C f_1(z) dz = 2\pi i \left(\frac{1}{2} + \frac{e^{-2}}{2i} \left(\frac{e^{2i}}{-2i} + \frac{e^{-2i}}{-2i} \right) \right)$$

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$$= 2\pi i \left(\frac{1}{2} + \frac{e^{-2}}{4} \cdot 2 \cos 2 \right)$$

$$= \pi i (1 + e^{-2} \cos 2)$$

Let

$$I_2 = \int_C f_2(z) dz, \quad \because f_2(z) \text{ is analytic inside & on } C$$

$$\therefore \int_C f_2(z) dz = 0 \quad (\text{By Cauchy's theorem})$$

$$I_3 = \int_C f_3(z) dz = \int_C \frac{1}{(z-4)^2} dz$$

$f_3(z)$ is analytic inside and on C

$$\therefore \text{By Cauchy's theorem, } I_3 = 0$$

$$\therefore \int_C f_1(z) + f_2(z) + f_3(z) dz = \pi i (1 + e^{-2} \cos 2)$$

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(8e)

Let $f(z)$ be a function with no singularity in finite and infinite part.

$\therefore f(z)$ has no singularity at finite
 $\therefore$ Expand $f(z)$ along $z=0$ with Laurent Series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots \quad \text{--- (I)}$$

there will be no principle part

Now, consider $f(1/z)$, it also will have no principle part as there is no singularity at infinity

Let $w = 1/z$, Laurent's series along $w=0$

$$f(w) = b_0 + b_1 w + b_2 w^2 + \dots \quad b_n w^n$$

$$f(z) = b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots \quad \text{--- (II)}$$

But, $\therefore$ both function in (I) and (II) are same $\therefore$ By equating coefficient

of $z, 1/z$ and so on -

$$a_1 = 0 \quad b_1 = 0, \dots$$

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similarly all coefficient of z^n or $\frac{1}{z^n}$ will be 0

$$\therefore f(z) = a_0 = b_0$$

i.e. $f(z)$ will be constant

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7a

Let K be a finite fieldTo show no. of element in K is p^n i.e. $|K| = p^n$
 p is prime & characteristicWe know that, K is field $\Rightarrow K$ is Integral domain

and characteristic of every I.D is either 0 or prime.

as K is finite $\text{char}(K) \neq 0$. $\therefore$ claim: $\text{char}(K) = \text{prime}$ Proof: let $\text{char}(K) = p = mn$ be composite
no. $m, n \neq 1$ Now, ~~for~~ $\forall a \in K$, $a^2 \in K$

$$(mn)(a) = 0 \quad (\text{property of char})$$

$$\Rightarrow a^2 + a^2 + \dots + a^2 = 0$$

$$\Rightarrow \underbrace{(a + a + \dots + a)}_{m \text{ times}} \underbrace{(a + a + \dots + a)}_{n \text{ times}} = 0$$

$$\Rightarrow (ma)(na) = 0$$

$$\therefore K \text{ is I.D} \Rightarrow ma = 0 \text{ or } na = 0$$

In any case $m < p$ & $n < p$ $\therefore \forall a \in K \exists m \text{ or } n$ which isless than p & $ma = 0$ or $na = 0$
 $\Rightarrow p = mn$ is not characteristic

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$\therefore$ finite I.D has prime character

Now, consider any element $a \in K$

let p be prime s.t $\text{char}(a) = p$

Now, Consider Subgroup generated by a

$$\langle a \rangle = \{ a, 2a, \dots, (p-1)a, 0 \}$$

$\therefore p$ is prime $\therefore$ there will p element in $\langle a \rangle$

$\therefore \forall$ element $a \in K$, $\langle a \rangle$ will have p element

By Lagrange's theorem, $o(\langle a \rangle) \mid o(K) \quad \forall a \in K$

$$\Rightarrow p \mid o(K)$$

$\therefore o(K)$ will be multiples of p^n

$\therefore o(K) = p^n$ for some $n \geq 1$

Consider $G = \frac{\mathbb{Z}_3[X]}{(x^2+1)}$, where (x^2+1) is
ideal of $\mathbb{Z}_3[X]$

To prove: G is field

Consider, $\alpha, \beta \in G$ s.t

$$\alpha = (x^2+1) + ax + b \quad \beta = (x^2+1) + cx + d$$

s.t $ax+b \notin (x^2+1)$

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Consider, any $\beta = (x^2+1) + cx+d$

$$\text{Now, } \alpha\beta = (x^2+1) + (ax+b)(cx+d)$$

$$= (x^2+1) + acx^2 + (bc+ad)x + bd$$

$$= (x^2+1) + ac(x^2+1) - ac + bd + x(bc+da)$$

$$= (x^2+1) + x(bc+da) + (bd-ac)$$

$$= (x^2+1) + 1$$

$$\text{if } bc+ad=0 \quad \text{and } bd-ac=1$$

$$\Rightarrow c = -\frac{ad}{b}$$

$$\Rightarrow bd + a\left(\frac{ad}{b}\right) = 1$$

$$\Rightarrow d\left(\frac{a^2+b^2}{b}\right) = 1$$

$$\Rightarrow d = \frac{b}{a^2+b^2} \quad \& \quad c = -\frac{a}{a^2+b^2}$$

$$\therefore a, b \in \mathbb{Z}_3 \quad a, b = 0, 1, 2$$

then

a^2+b^2	0	1	2
0	0	1	1
1	1	2	2
2	1	2	2

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Number of element in $\frac{\mathbb{Z}_3[x]}{(x^2+1)}$

any element will be of form $(x^2+1) + ax+b$

a, b will have 3 values

$$\therefore \text{total elements} = 3 \times 3 = \boxed{9}$$

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(7b)

Given $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$

To show $S'(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 (1+n^2)^2} \neq x$

let $f_n(x) = \frac{1}{n^3 + n^4 x^2}$

Now, Consider, $|f_n(x)| = \frac{1}{|n^3 + n^4 x^2|} \leq \frac{1}{n^3} = M_n$ (say)

& $\sum M_n = \sum \frac{1}{n^3}$ is convergent
(by p-test where $p > 1$)

$\therefore$ By Weierstrass M-test, $\sum f_n(x)$ Converges.
uniformly

Now, $f_n'(x) = \frac{-1(2n^4 x)}{(n^3 + n^4 x^2)^2} = \frac{-2x n^4}{n^6 (1+n^2)^2}$
 $= \frac{-2x}{n^2 (1+n^2)^2}$

Consider, $|f_n'(x)| = \left| \frac{2x}{n^2 (1+n^2)^2} \right|$

Consider $y = \frac{2x}{n^2 (1+n^2)^2}$

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$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1}{n^2} \left[\frac{2(1+nx^2)^2 - 2x(2)(1+nx^2)(2nx)}{(1+nx^2)^4} \right] \\
 &= \frac{1}{n^2} \left[\frac{(1+nx^2)(2+2nx^2-8nx^2)}{(1+nx^2)^4} \right] \\
 &= \frac{2(1-3nx^2)}{n^2(1+nx^2)^3} = 0 \text{ when } x = \frac{1}{\sqrt{3}n}
 \end{aligned}$$

$$\frac{d^2y}{dx^2} = \frac{2}{n^2} \left[\frac{-6nx(1+nx^2)^3 - 3(1+nx^2)^2(2nx)}{(1+nx^2)^6} \right]$$

$$\text{at } x = \frac{1}{\sqrt{3}n} \quad \frac{d^2y}{dx^2} = \frac{2}{n^2} \cdot \frac{\left(-\frac{6n}{\sqrt{3}n} \left(1+\frac{1}{3}\right)^3\right)}{\left(1+\frac{1}{3}\right)^6} < 0$$

$$\therefore y \text{ is max}^m \text{ at } x = \frac{1}{\sqrt{3}n}$$

$$\therefore |f_n'(x)| \leq \left| \frac{2/\sqrt{3}n}{n^2(4/3)^2} \right|$$

$$= \frac{9 \cdot 2}{16 \cdot \sqrt{3} n^2 \sqrt{n}}$$

$$\Rightarrow |f_n'(x)| \leq \frac{9}{8\sqrt{3}} n^{5/2} = M_n$$

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$\sum M_n$ is Convergent by p-test
as $p = \frac{1}{2} > 1$

$\therefore$ By Weierstrass M-test $\sum f_n'(x)$ Converges uniformly

$\therefore$ We can show that,

$$S'(x) = \frac{d}{dx} \left(\sum f_n(x) \right)$$

$$= \sum \frac{d}{dx} (f_n(x))$$

$$= \sum f_n'(x)$$

$$= \sum \frac{-2x}{n^2(1+x^2)^2}$$

$\forall x$.

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(7c)

$$\text{Let } f(z) = \frac{z^3}{(z-1)(z-2)(z-3)} \quad \text{at } z=1,$$

$f(z)$ is analytic except at $z=1, 2, 3$

By residue theorem, $\therefore z=1$ has simple pole

$$\begin{aligned} \therefore \text{Residue of } f(z) \text{ at } (z=1) &= \lim_{z \rightarrow 1} \frac{z^3}{(z-2)(z-3)} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{Residue of } f(z) \text{ at } (z=2) &= \lim_{z \rightarrow 2} \frac{z^3}{(z-1)(z-3)} \\ &= -8 \end{aligned}$$

$$\begin{aligned} \text{Residue of } f(z) \text{ at } (z=3) &= \lim_{z \rightarrow 3} \frac{z^3}{(z-1)(z-2)} \\ &= \frac{27}{2} \end{aligned}$$

We know that, residue of f at infinity is

given by residue of $-\frac{1}{z^2} f(1/z)$ at $z=0$

Residue of $-\frac{1}{z^2} f(1/z) \rightarrow$

$$f(1/z) = \frac{1}{z^3} \frac{1}{(1-z)(1-2z)(1-3z)}$$

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$$f(1/2) = \frac{-1}{(z-1)(2z-1)(3z-1)}$$

$$\text{Now, } -\frac{1}{z^2} f(1/2) = \frac{1}{z^2 (z-1)(2z-1)(3z-1)}$$

$$\text{Let } g(z) = \frac{1}{z^2} f(1/2)$$

$$\text{residue of } g(z) \text{ at } (z=0) = \lim_{z \rightarrow 0} \frac{d}{dz} \left(\frac{1}{(z-1)(2z-1)(3z-1)} \right)$$

$$= \lim_{z \rightarrow 0} \frac{(-1) \left((2z-1)(3z-1) + 2(z-1)(3z-1) + 3(z-1)(2z-1) \right)}{(z-1)^2 (2z-1)^2 (3z-1)^2}$$

$$= -\frac{1}{1} (1 + 2 + 3) = -6$$

Now, sum of all residues

$$= \frac{1}{2} - 8 + \frac{27}{2} - 6 = \boxed{10}$$

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(86)

$$\text{let } I = \int_0^{\infty} \frac{x \sin mx}{a^2 + x^2} dx \left(= \frac{\pi}{2} e^{-ma} \right) \text{ to prove.}$$

Consider ~~f(z)~~ = z

$$I = \int_0^{\infty} \frac{x \sin mx}{a^2 + x^2} dx = \text{Imaginary part of } \int_0^{\infty} \frac{x e^{imx}}{a^2 + x^2} dx$$

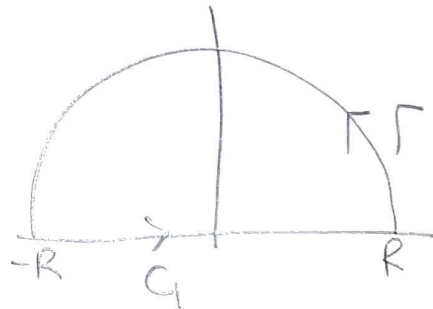
$$\text{let } f(z) = \frac{ze^{imz}}{a^2 + z^2} \quad \& \quad I' = \int_C f(z) dz, \text{ where,}$$

and consider contour C as given below,

$$C = \Gamma + G$$

$$\text{where } \Gamma: z = Re^{i\theta} \quad \theta = 0 \text{ to } \pi$$

$$G: z = t \quad t = -R \text{ to } R$$

Now, singular point of $f(z)$ are

$$z^2 + a^2 = 0 \Rightarrow z = \pm ai$$

 $\therefore$ only $z = ai$ lie inside C $\therefore$ By residue theorem,

$$\int_C f(z) dz = 2\pi i (\text{residue at } z = ai)$$

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$$\begin{aligned} \text{Residue at } (z=ai) &= \lim_{z \rightarrow ai} \frac{z e^{imz}}{(z+ai)} \\ &= \frac{ai e^{-ma}}{2ai} = \frac{e^{-ma}}{2} \end{aligned}$$

$$\therefore \int_C f(z) dz = 2\pi i \times \frac{e^{-ma}}{2} = i \left(\frac{\pi e^{-ma}}{1} \right)$$

Now, Consider $\int_{\Gamma} f(z) dz$

$$\therefore \lim_{z \rightarrow \infty} \frac{z}{a^2 + z^2} = 0$$

$$\therefore \text{By Jordan's lemma, } \int_{\Gamma} \frac{e^{imz} z}{a^2 + z^2} dz = 0$$

$$\therefore \int_C f(z) dz = \int_{\Gamma} f(z) dz = i \pi \frac{e^{-ma}}{2}$$

$$\Rightarrow \int_{-R}^R \frac{x e^{imx}}{a^2 + x^2} dx = i \pi \frac{e^{-ma}}{2}$$

$$\text{as } R \rightarrow \infty \quad \int_0^{\infty} \frac{x e^{imx}}{a^2 + x^2} dx = i \pi \frac{e^{-ma}}{2}$$

$$\therefore \int_0^{\infty} \frac{x \sin mx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ma}$$

Imaginary part of above equation.

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8c

$$\text{let } f(z) = z^8 - 4z^5 + z^2 - 1$$