

NEXT IAS

(To be filled by candidate)

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Date of Examination : 24/7

Exam Centre : Old Rajinder Nagar ☐

Online ☒

CSE (MAINS) TEST SERIES-2024

Test-02

MATHEMATICS
OPTIONAL

Dated : 30-06-2024

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are EIGHT questions divided in Two sections and printed in ENGLISH.

Candidate has to attempt FIVE questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

DELHI CENTRE:

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	20			
2. (b)	18	15			
2. (c)	21	15			
3. (a)	24	20			
3. (b)	27	15			
3. (c)	30	15			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	40	20			
5. (a)	44	10			
5. (b)	45	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	57	15			
6. (c)	61	15			
7. (a)	64	20			
7. (b)	67	15			
7. (c)	70	15			
8. (a)	74	20			
8. (b)	77	15			
8. (c)	81	15			
Grand Total					

Remarks

Observations:

1. When the quantity of the substance is increased, the rate of reaction also increases. This is because there are more particles available to collide and react.

2. The rate of reaction is directly proportional to the concentration of the reactants. If the concentration of the reactants is doubled, the rate of reaction also doubles.

3. The rate of reaction is also affected by the temperature. As the temperature increases, the rate of reaction increases. This is because the particles have more kinetic energy and move faster, leading to more frequent and energetic collisions.

4. The rate of reaction is also affected by the surface area of the solid reactants. If the surface area is increased, the rate of reaction increases. This is because there are more particles exposed to the other reactant, leading to more collisions.

(10/10/20)

21, 2.0, 1.0 (11/12)

1.0, 2.0, 1.0 (11/12)

SECTION-A

Q.1 (a) Show that every group of order 15 is cyclic.

(10)

let G be a group of order 15

~~Case I~~ let $a \neq e \in G$

Case I: If $o(a) = 15$

then let $H = \langle a \rangle$

$$o(H) = 15$$

$$H \leq G$$

H is cyclic subgroup of G

$$\text{but } o(H) = o(G) \Rightarrow G = H$$

$\therefore G$ is cyclic

Case II. If $a \in G$, $o(a) \neq 15$ and $H = \langle a \rangle$
and $a \neq e$.

~~Let~~ H is a subgroup of G

By Lagrange theorem (as G is finite)

$$o(H) \mid o(G)$$

$$\Rightarrow o(H) = 1, 3, 5, 15$$

$$o(H) \neq 1 \text{ as } a \neq e.$$

Q.1 (b)

Show that the smallest subgroup V of A_4 containing $(1, 2)(3, 4)$, $(1, 3)(2, 4)$ and $(1, 4)(2, 3)$ is isomorphic to Klein 4-group. (10)

$$\text{Klein 4 group} = \{a, b, c, e\}$$

where each element is its self inverse

$$\text{i.e. } a^2 = b^2 = c^2 = e^2 = e$$

$$V = \{(1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3), e\}$$

$V = \{a_1, a_2, a_3, e\}$
In isomorphism, identities are mapped to identities and order of pre image and image is same.

let $f: V \rightarrow K_4$ is isomorphism

$$o((1, 2)(3, 4)) = \text{lcm}(2, 2)$$

$$= 2 \quad (\text{as } (1, 2), (3, 4) \text{ are disjoint})$$

$$\text{similarly } o[(1, 3)(2, 4)] = o[(1, 4)(2, 3)] = 2$$

$$\begin{aligned} \text{also } [(1, 2)(3, 4)]^{-1} &= (3, 4)^{-1} (1, 2)^{-1} \\ &= (4, 3)(2, 1) \\ &= \end{aligned}$$

Candidates
must not write
on this margin

So there

$$f(a_1) = a$$

$$f(a_2) = b$$

$$f(a_3) = c$$

$$f(e_v) = e$$

$$f(a_1) = \cancel{a} = c$$

$$f(a_2) = a$$

$$f(a_3) = b$$

$$f(e_v) = e$$

$$f(a_1) = b$$

$$f(a_2) = c$$

$$f(a_3) = a$$

$$f(e_v) = e$$

$$f(a_1) = c$$

$$f(a_2) = b$$

$$f(a_3) = a$$

$$f(e_v) = e$$

$$f(a_1) = a$$

$$f(a_2) = c$$

$$f(a_3) = b$$

$$f(e_v) = e$$

$$f(a_1) = b$$

$$f(a_2) = a$$

$$f(a_3) = c$$

$$f(e_v) = e$$

are 6 isomorphism.

1 (c) If $\sum u_n$ converges, prove that $\sum \frac{u_n}{1-u_n}$, ($u_n \neq 1$) also converges.

(10)

$\sum u_n$ converges

$$\Rightarrow \lim_{n \rightarrow \infty} u_n = 0$$

$$\text{i.e. } |u_n - 0| < \epsilon \quad \forall n > M$$

$$|u_n| < \epsilon$$

$$-|u_n| > -\epsilon$$

$$1 - |u_n| > 1 - \epsilon$$

$$\frac{1}{1 - |u_n|} < \frac{1}{1 - \epsilon}$$

$$\frac{1}{1 + \epsilon} \leq \frac{1}{1 + |u_n|} \leq \frac{1}{1 - |u_n|} \leq \frac{1}{1 - \epsilon}$$

$$\frac{u_n}{1 + \epsilon} < \frac{u_n}{1 - u_n} < \frac{u_n}{1 - \epsilon}$$

$$\frac{\sum u_n}{1 + \epsilon} < \sum \frac{u_n}{1 - u_n} < \frac{1}{1 - \epsilon} \sum u_n \quad \forall n > m$$

where $\epsilon \rightarrow 0$

as $\sum u_n$ converges

$$\Rightarrow \frac{1}{1 - \epsilon} \sum u_n \text{ also converges}$$

where $\epsilon \rightarrow 0$

$(\because$ as $\sum a_n$ converges $\Rightarrow c \sum a_n$ converges)

$$\Rightarrow \sum \frac{u_n}{1-u_n} \text{ converges } \forall n > m$$

(by comparison test)

Also $\sum_{n=1}^m \frac{u_n}{1-u_n} = K$

$$\Rightarrow \sum \frac{u_n}{1-u_n} = K + \sum_{n>M} \frac{u_n}{1-u_n}$$

RHS is convergent
 $\therefore$ LHS is convergent.

- Q.1 (e) Obtain the first three terms of Laurent Series expansion of the function $f(z) = \frac{1}{e^z - 1}$ about the point $z = 0$ valid in the region $0 < |z| < 2\pi$. (10)

$$e^z$$

$g(z) = e^z$ is analytic in $0 < |z| < 2\pi$
so expanding by Taylor's series about $z = 0$

$$\begin{aligned} g(z) &= g(0) + \sum_n \frac{g^{(n)}(0)}{n!} (z-0)^n \\ &= 1 + \sum_n \frac{z^n}{n!} \end{aligned}$$

$$f(z) = \frac{1}{g(z) - 1} = \frac{1}{\sum_n \frac{z^n}{n!}}$$

$$f(z) = \frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots}$$

$$= \frac{1}{z \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots \right)}$$

$$= \frac{1}{z} \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots \right)^{-1}$$

$$= \frac{1}{z} \left(1 - \left(\frac{z}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots \right) \right)$$

$$= \frac{1}{z} \left(1 - \frac{z}{2!} + \frac{z^2}{(2!)^2} - \frac{z^3}{3!} + \dots \right)$$

$$= \frac{1}{z} - \frac{1}{2!} + \frac{z}{2^2} - \frac{z^2}{6} + \dots$$

- 2 (a) Let S be the set of all real numbers except -1 . Define $*$ on S by $a * b = a + b + ab$. Is $(S, *)$ a group? Find the solution of the equation $2 * x * 3 = 7$ in S . (20)

To check $S \neq (S, *)$ is group

1) Closure $a, b \in S$
 $a * b = a + b + ab$

$$a, b \in \mathbb{R} \Rightarrow a + b + ab \in \mathbb{R} - i$$

$$a + b + ab = -1$$

$$\Rightarrow (a+1)(b+1) = 0$$

$$\Rightarrow a = -1, b = -1$$

$$\Rightarrow a \text{ but } a, b \in S$$

$$\Rightarrow a + b + ab \neq -1 \text{ --ii--}$$

$$\Rightarrow a + b + ab \in S \text{ (i, ii)}$$

2) Associative

$$a * (b * c) = a * (b + c + bc) = a + (b + c + bc) + a(b + c + bc)$$

$$= a + b + c + b(c) + a(b) + a(c) + a(bc)$$

(by associativity of $\mathbb{R}$)

$$= [a + b + a(b)] + [c] + (a + b + ab)c$$

(by ~~commutativity~~ ^{associativity} of

real numbers

multiplication

$$= (a+b+ab)^n c$$

$$= (a^n b^n) c$$

3) Identity

$$a^n b = a \Rightarrow a + b + ab = a$$

$$\Rightarrow b + ab = 0$$

$$\Rightarrow b(1+a) = 0 \quad (\text{cancellation in real numbers})$$

$$\Rightarrow b = 0 \text{ is right identity (as } a \neq -1 \text{ as } a \in S)$$

4) Inverse.

$$a^n b = 0 \Rightarrow a + b + ab = 0$$

$$\Rightarrow b(1+a) = -a$$

$$b = \frac{-a}{1+a}$$

$$\text{If } b = -1 \quad \frac{-a}{1+a} = -1 \Rightarrow a = 1+a \Rightarrow 0 = 1 \text{ absurd}$$

$$\therefore b \neq -1$$

$$\Rightarrow b \in S$$

$$\Rightarrow \frac{-a}{1+a} \text{ is right inverse}$$

Hence S is group.

$$2^n x^n 3 = 2^n (x^n 3) = 2^n (x + 3 + 3x)$$

$$= 2 + (x + 3 + 3x) + 2(x + 3 + 3x) = 7$$

$$12x + 11 = 7 \Rightarrow x = -1/3.$$

Q.2 (b) If $\langle a_n \rangle$ is a sequence of positive real numbers such that $a_n = \frac{1}{2}(a_{n-1} + a_{n-2}) \forall n \geq 3$, then show that $\langle a_n \rangle$ converges to $\frac{1}{3}(a_1 + 2a_2)$. (15)

$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$$

$$a_3 = \frac{1}{2}(a_2 + a_1)$$

$$a_4 = \frac{1}{2}(a_3 + a_2) = \frac{1}{2}\left(\frac{1}{2}(a_2 + a_1) + a_2\right) = \frac{1}{2^2}(3a_2 + a_1)$$

$$a_5 = \frac{1}{2}(a_4 + a_3) = \frac{1}{2}\left(\frac{1}{2^2}(3a_2 + a_1) + \frac{1}{2}(a_2 + a_1)\right) = \frac{1}{2^3}(5a_2 + 3a_1)$$

$$a_6 = \frac{1}{2}(a_5 + a_4) = \frac{1}{2}\left(\frac{1}{2^3}(5a_2 + 3a_1) + \frac{1}{2^2}(3a_2 + a_1)\right) = \frac{1}{2^4}(11a_2 + 5a_1)$$

$$a_n - a_{n-1} = \frac{1}{2}(a_{n-1} + a_{n-2}) - a_{n-1} = \frac{a_{n-2} - a_{n-1}}{2}$$

$$a_7 = \frac{1}{2}(a_6 + a_5) = \frac{1}{2}\left(\frac{1}{2^4}(11a_2 + 5a_1) + \frac{1}{2^3}(5a_2 + 3a_1)\right) = \frac{1}{2^5}(21a_2 + 11a_1)$$

$$a_8 = \frac{1}{2}(a_7 + a_6) = \frac{1}{2}\left(\frac{1}{2^5}(21a_2 + 11a_1) + \frac{1}{2^4}(11a_2 + 5a_1)\right) = \frac{1}{2^6}(43a_2 + 21a_1)$$

$$a_n - a_{n-1} = \frac{a_{n-2} - a_{n-1}}{2}$$

$$a_{n+1} - a_{n-2} = \frac{a_{n-3} - a_{n-2}}{2}$$

⋮

$$a_3 - a_2 = \frac{a_1 - a_2}{2}$$

Adding

$$a_n - a_2 = \frac{a_1 - a_{n-1}}{2}$$

$$2a_n + \frac{a_{n-1}}{2} = a_1 + 2a_2 \quad \text{--- (1)}$$

$$a_n - a_{n-1} = \frac{a_{n-2} - a_{n-1}}{2}$$

And

$$a_{n+1} - a_{n-2} = \frac{a_{n-3} - a_{n-2}}{2}$$

$$\Rightarrow a_n - a_{n-1} = - \frac{(a_{n-3} - a_{n-2})}{2^2}$$

$$|a_n - a_{n-1}| = \frac{1}{2^2} |a_{n-3} - a_{n-2}|$$

Continuing

$$|a_n - a_{n-1}| = \frac{1}{2^{n-1}} |a_2 - a_1|$$

~~$$|a_n - a_{n-1}| = \frac{1}{2^{n-1}} |a_2 - a_1|$$~~

$$|a_n - a_m| = |a_n - a_{n-1} + a_{n-1} - a_{n-2} + a_{n-2} - \dots - a_m|$$

$$\leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \dots + |a_{m+1} - a_m|$$

$$\leq \frac{1}{2^{n-1}} |a_2 - a_1| + \frac{1}{2^{n-2}} |a_2 - a_1| + \dots + \frac{1}{2^m} |a_2 - a_1|$$

$$\leq |a_2 - a_1| \left[\frac{1}{2^{n+1}} + \dots + \frac{1}{2^m} \right]$$

$$\leq |a_2 - a_1| \left[\frac{1}{2^{n-1}} \right] \left[1 + 2 + 2^2 + \dots + 2^{m-n} \right]$$

$$\leq |a_2 - a_1| \left[\frac{1}{2^{n-1}} \right] \left[\frac{2^{m-n+1} - 1}{2 - 1} \right]$$

$$|a_n - a_m| \leq |a_2 - a_1| \left[2^{1-m} - 2^{1-n} \right]$$

$$|a_n - a_m| \leq |a_2 - a_1| 2^{1-m} < \epsilon$$

$$\text{i.e. } 2^{1-m} < \frac{\epsilon}{|a_2 - a_1|}$$

$$(1-m) \log 2 < \frac{\log \epsilon}{\log |a_2 - a_1|}$$

$$m > 1 - \frac{\log \epsilon}{\log 2 (a_2 - a_1)}$$

$$|a_n - a_m| < \epsilon \quad \forall \quad m > 1 - \frac{\log \epsilon}{\log 2 (a_2 - a_1)}$$

~~$\Rightarrow a_n$ is convergent~~

$\Rightarrow a_n$ is Cauchy sequence $\Rightarrow$ convergent

① $\Rightarrow$

$$\lim_{n \rightarrow \infty} a_n = a \quad (a \text{ is convergent})$$

$$\Rightarrow 3a = a_1 + 2a_2$$

$$\Rightarrow a = \frac{1}{3}(a_1 + 2a_2)$$

Solve graphically the following LPP:

$$\text{Max } z = 5x_1 - 3x_2$$

subject to

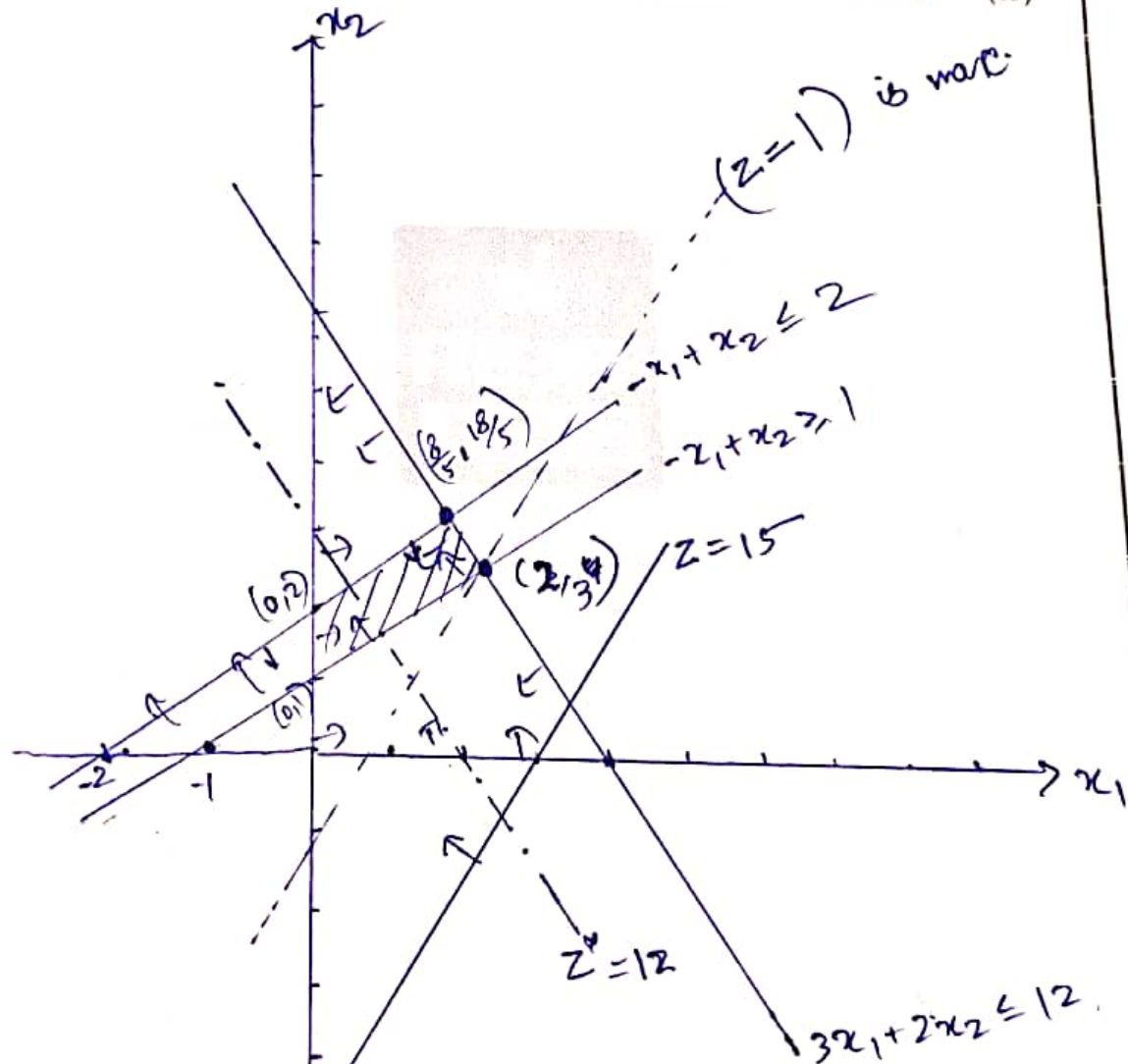
$$3x_1 + 2x_2 \leq 12$$

$$-x_1 + x_2 \geq 1$$

$$-x_1 + x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

If the objective function is changed to $\text{Max } z = 6x_1 + 4x_2$, while the constraints remain the same, then comment on the number of solutions. Will $(4, 0)$ be also a solution? (15)



$$Z = 5x_1 - 3x_2 = 15$$

$$\frac{x_1}{3} - \frac{x_2}{5} = 1$$

$$Z = 1 \text{ is max at } (x_1, x_2) = (2, 3)$$

Case II $\sum: 6x_1 + 4x_2 = 12$
 $\Rightarrow \underline{x_1} + \underline{x_2} = 1$

Z^* will have infinite solutions
lying on line $3x_1 + 2x_2 = 12$

$(4, 0)$ will also be a solution



Q.3 (a)

Let M be a subgroup and N a normal subgroup of a group G . show that MN is a subgroup of G and MN/N is isomorphic to $M/(M \cap N)$.

(20)

$$M \leq G, \quad N \trianglelefteq G.$$

To prove $MN \leq G$

$$\text{let } m_1 n_1 \in MN$$

$$m_2 n_2 \in MN$$

$$\begin{aligned} & (m_1 n_1)(m_2 n_2)^{-1} \\ &= (m_1 n_1)(n_2^{-1} m_2^{-1}) \end{aligned}$$

$$= (m_1)(n_1 n_2^{-1}) m_2^{-1} \quad (\text{by associativity of } G)$$

$$= (m_1)(n_3 m_2^{-1}) \quad (n_1 n_2^{-1} \in N \text{ as } N \trianglelefteq G \Rightarrow N \leq G)$$

$$= (m_1)[n_3 m_2^{-1} n_3^{-1} n_3]$$

$$= m_1 [n_3 m_2^{-1} n_3^{-1}] n_3$$

$$= m_1 [n_4 n_3] \quad \left(\begin{array}{l} \text{as } n_3 m_2^{-1} n_3^{-1} \in N \\ \text{as } m_2 \in M \subset G \end{array} \right)$$

$$m_1 (n_5) = m_1 n_5 \quad (n_4 n_3 \in N \text{ closure})$$

$$\in MN$$

$$\Rightarrow (m_1 n_1)(m_2 n_2)^{-1} \in MN$$

$$\Rightarrow MN \leq G.$$

To prove $N \trianglelefteq MN$; Clearly $N \subseteq MN$ and $N \neq \emptyset$

$$m_1 n_1 \in MN \neq n_2 \in N$$

consider $(m_1 n_1) (n_2) (m_1 n_1)^{-1}$

$$= (m_1 n_1) (n_2) (n_1^{-1} m_1^{-1})$$

$$= n_1 (n_1^{-1} m_1 n_1) (n_3 m_1 n_3^{-1} n_3)$$

where $n_3 = n_2 n_1^{-1} \in N$ (by closure)

$$= n_1 (n_4) (n_5) n_3 \in N$$

as $m_1 \in M \subseteq G$

$$\Rightarrow n_1 (n_4) (n_5) n_3 \in N$$

$$\Rightarrow N \trianglelefteq MN$$

$$\Rightarrow \frac{MN}{N} \text{ is defined.}$$

consider $f: M \rightarrow \frac{MN}{N}$

$$f(m) = mN$$

well defined

$$m_1, m_2 \in M$$

$$m_1 = m_2$$

$$\Rightarrow m_1 N = m_2 N$$

$$\Rightarrow f(m_1) = f(m_2)$$

One One

$$f(m_1) = f(m_2) \Rightarrow m_1 N = m_2 N$$

$$\Rightarrow m_1 m_2^{-1} \in N$$

$$\Rightarrow \# / m_1 m_2$$

$$f(m_1 m_2) = N m_1 m_2$$

$$\text{let } m_1 N \in \frac{MN}{N}$$

$$\Rightarrow m_1 N \in \frac{MN}{N}$$

$$\Rightarrow m_1 \in M$$

$$\Rightarrow f(m_1) = m_1 N =$$

$\Rightarrow f$ is onto

By homomorphism theorem

$$\frac{MN}{N} \cong \frac{M}{\text{Ker } f}$$

$$\text{Ker } f = \{m \mid f(m) = e\}$$

e is identity of $\frac{MN}{N}$

$$\text{let } m \in M \quad \text{i.e. } N$$

$$f(m) = N$$

$$mN = N$$

$$\Rightarrow m \in N$$

$$\text{Also } m \in M \text{ (domain)}$$

from 1, 2

$$m \in N \cap M \quad \text{i.e. } \text{Ker } f = MN$$

$$\Rightarrow \frac{MN}{N} \approx \frac{M}{MNN}$$

any other value

is not

the thing is that

balance is not

balance is not

(I am) Every thing is not

(I am) Every thing is not

is not

any kind of "yes" or "no" is not

is not

any kind of "yes" or "no" is not

the thing is that

is not

Q.3 (b) Every bounded sequence has at least one limit point.

(15)

let $\langle a_n \rangle$ be a bounded sequence

$$|a_n| < K \quad \forall n \in \mathbb{N}$$

Case I let $S: \{a_n\}$ is infinite set

As $\{a_n\}$ is bounded

$\Rightarrow S$ is bounded

So S has a limit point (say l_1)

(Bolzano Weierstrass Theorem).

S is infinite and bounded.

S has infinite elements in nbd of l_1 (by defⁿ of limit point)

$\Rightarrow \langle a_n \rangle$ has infinite terms in nbd of l_1

$\therefore \langle a_n \rangle$ has a limit point

Case $S: \{a_n\}$ is finite set.

$\therefore \langle a_n \rangle$ has infinite terms in nbd of l .

$\therefore l$ ~~has~~ is limit point.

3 (c)

If all zeros of a polynomial lies in a half plane then show that zeros of derivative $P'(z)$ also lies in the same half plane.

(15)

$$P(z) = \prod_j (z - \alpha_j)$$

without loss of generality
let $\alpha_{yj} > 0$

i.e.

zeros lies in upper half plane. Now let zeros of $P'(z)$ are given by $\beta_{xj} + i\beta_{yj}$ And let $\beta_{yj} < 0$

$$\frac{P'(z)}{P(z)} = \sum_j \frac{1}{z - \alpha_j} = \sum_j \frac{1}{z - (\alpha_{xj} + i\alpha_{yj})}$$

$$P'(z) = P(z) \sum_j \frac{1}{z - (\alpha_{xj} + i\alpha_{yj})}$$

$$P'(z_p) = P(z_p) \sum_j \frac{1}{(\beta_{xj} - \alpha_{xj}) + i(\beta_{yj} - \alpha_{yj})}$$

$$P'(z_p) = P(z_p) \sum_j \frac{(\beta_{xj} - \alpha_{xj}) - i(\beta_{yj} - \alpha_{yj})}{\sqrt{(\beta_{xj} - \alpha_{xj})^2 + (\beta_{yj} - \alpha_{yj})^2}}$$

$$P'(z_p) = P(z_p) \sum_j \frac{\beta_{xj} - \alpha_{xj}}{(\beta_{xj} - \alpha_{xj})^2 + (\beta_{yj} - \alpha_{yj})^2} - i P(z_p) \sum_j \frac{\beta_{yj} - \alpha_{yj}}{(\beta_{xj} - \alpha_{xj})^2 + (\beta_{yj} - \alpha_{yj})^2}$$

$= 0$ ($P(z_p) \neq 0$) as z_p lies in lower half plane but

$$\Rightarrow \sum_j \frac{\beta_{yj} - \alpha_{yj}}{(\beta_{xj} - \alpha_{xj})^2 + (\beta_{yj} - \alpha_{yj})^2} = 0$$

zeros of $P(z)$ are in upper half

$$-\sum_j \frac{(|\beta_{yj}| + |\alpha_{yj}|)}{(\beta_{yj} - \alpha_{yj})^2 + (|\beta_{yj}| + |\alpha_{yj}|)^2} = 0$$

This is never possible as Denominator is always positive and Numerator is always negative for each term of summation

$\therefore$ Hypothesis is contradicted

$\therefore$ All ~~roots~~ $\beta_{yj} > 0$

Hence zeros of $P(z)$ and $P'(z)$ lie in same half plane.

SECTION-B

Let R be a non zero-commutative ring with unity. If every ideal of R is prime, prove that R is a field. (10)

let $a \neq 0 \in R$
 let $I = a^2R = \{a^2r \mid r \in R\}$
 be an ideal.

To prove I is ideal.

① $a^2r_1, a^2r_2 \in I$

$$a^2(r_1) - a^2r_2 = a^2(r_1 - r_2) \in a^2R$$

as $r_1 - r_2 \in R$

② let $r_1 \in R$

$$\begin{aligned} r_1(a^2r) &= (r_1a^2)r \quad (\text{by associativity of } R) \\ &= (a^2r_1)r \quad (\text{by commutativity of } R) \\ &= a^2(r_1r) \quad (\text{by associativity of } R) \\ &= a^2r_2 \quad (r_1r \in R) \end{aligned}$$

$\Rightarrow r_1(a^2r) \in a^2R$
 $\Rightarrow I$ is left ideal

③ let $r_3 \in R$

$$\begin{aligned} (a^2r)r_3 &= a^2(r r_3) \quad (\text{by associativity of } R) \\ &= a^2r_4 \quad (r r_3 \in R) \\ &\in a^2R \end{aligned}$$

$\Rightarrow I$ is right ideal

$\Rightarrow I$ is ideal.

Given I is prime

Let $x_1 \in R$

an $a^2 x_1 \in I$; $a^2 x_1 = a(ax_1)$

$\Rightarrow$ either $a \in I$ or $x_1 \in I$
because I is prime ideal

Case I $a \in I$

$\Rightarrow \nexists x_2 \in R$

such that $a x_2 = a$

$\Rightarrow a x_2 - a = 0$

$\Rightarrow a(ax_2 - 1) = 0$ ($\because 1 \in R$)

~~$\Rightarrow ax_2 - 1 = 0$ (as $a \neq 0$ and~~

To show R is integral domain.

let $a, b \in R$ and $ab = 0$

$ab = 0 \Rightarrow$ either $a \in \{0\}$
or $b \in \{0\}$

(as $\{0\}$ is ideal of R and its given that every ideal is prime)

$\Rightarrow$ either $a = 0$ or $b = 0$

$\Rightarrow R$ is integral domain.

$\Rightarrow a(ax_2 - 1) = 0$

$\Rightarrow ax_2 - 1 = 0$ (as $a \neq 0$ and R is without 0 divisors)

$\Rightarrow ax_2 = 1$

$\Rightarrow a$ has multiplicative inverse and commutative $x_2 a = 1$ hold

Q.5 (b)

Prove that if $u_n(x)$, $n = 1, 2, \dots$ are continuous in $[a, b]$ and if $\sum u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, $S(x)$ is continuous in $[a, b]$. (10)

$$\sum u_n(x) = S(x)$$

$$|\sum u_n(x) - S(x)| < \epsilon^1 \quad \text{where}$$

S is independent of x
i.e. $S = S(\epsilon^1)$

To prove

$$|S(x) - S(c)| < \epsilon \quad \text{whenever}$$

$$|x - c| < \delta$$

~~$|S(x) - S(c)|$~~

$$+ \sum u_n(x) \rightarrow S$$

$$|S(x) - \sum u_n(x) + \sum u_n(x) - \sum u_n(c) + \sum u_n(c) - S(c)|$$

$$< |S(x) - \sum u_n(x)| + |\sum u_n(x) - \sum u_n(c)|$$

$$+ |\sum u_n(c) - S(c)|$$

$$|S(x) - \sum u_n(x)| < \epsilon^1 \quad (\text{given})$$

$$|u_n(x) - u_n(c)| < \epsilon^1 \quad \text{whenever } |x - c| < \delta$$

as $u_n(x)$ is continuous

Also $|\sum u_n(c) - S(c)| < \epsilon'$

as $|\sum u_n(x) - S(x)| < \epsilon'$

where S is independent of x

$\therefore$ uniformly continuous
continuous at $x=c$ also

i.e. $|\sum u_n(c) - S(c)| < \epsilon'$

combining

$$|S(x) - S(c)| < 3\epsilon'$$

choose $\epsilon' = \epsilon/3$

we get $|S(x) - S(c)| < \epsilon$

whenever $|x - c| < \delta$

Q.5 (c)

Prove that $x_1 = 2, x_2 = 1, x_3 = 0$ is feasible solution to the following set of equations:

$$2x_1 - x_2 + 3x_3 = 3$$

$$-6x_1 + 3x_2 + 7x_3 = -9$$

Is the solution basic? Justify your answer. If the solution is not basic, reduce it to a basic feasible one. (10)

$$Ax = B$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -6 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -6 & 3 & 7 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + 3R_1} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 16 \end{bmatrix}$$

$$\rho(A) = 2$$

$\therefore$ At max 2 non zero x_i 's.

To check $(2, 1, 0)$ for feasibility

$$2(2) - 1(1) + 3(0) = 3 = 3$$

$$-6(2) + 3(1) + 7(0) = -9 = -9$$

Hence it is feasible.

for basic

$$\begin{bmatrix} 2 \\ -6 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 3 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 7 \end{bmatrix} x_3 = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ are dependent

as $\begin{bmatrix} 2 \\ -6 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} \therefore x_1, x_2$ not

constitute set of basic variables
Hence $(2, 1, 0)$ not basic solution

Consider

$$\begin{aligned} 2\lambda_1 - \lambda_2 + 3\lambda_3 &= 0 \\ -6\lambda_1 + 3\lambda_2 + 7\lambda_3 &= 0 \end{aligned}$$

2 equation, 3 variables

$\Rightarrow$ 1 free variable.

let $\lambda_1 = 1$

$$\Rightarrow 3\lambda_3 - \lambda_2 = -2$$

$$3\lambda_2 + 7\lambda_3 = 6$$

$$\lambda_2 = \frac{32}{23}, \lambda_3 = \frac{6}{23}$$

let $\frac{x_2}{\lambda_2} = \min \left\{ \frac{x_i}{\lambda_i} \mid \lambda_i > 0 \right\}$

$$= \min \left\{ \frac{2}{1}, \frac{23}{32} \right\}$$

$$= \frac{23}{32}$$

$$\hat{x}_1 = x_1 - \frac{23}{32} \lambda_1 = 2 - \frac{23}{32} = \frac{41}{32}$$

$$\hat{x}_2 = x_2 - \frac{23}{32} \lambda_2 = 1 - 1 = 0$$

$$\hat{x}_3 = x_3 - \frac{23}{32} \lambda_3 = -\frac{3}{16}$$

$\left\{ \frac{41}{32}, 0, -\frac{3}{16} \right\}$ is basic feasible solution

Q.5 (d) Evaluate $\int_C \left[\frac{e^{2z}}{z^2(z^2+2z+2)} + \log(z-6) + \frac{1}{(z-4)^2} \right] dz$ where C is the circle $|z|=3$. State the theorem you use in evaluating the above theorem. (10)

$$\int \frac{e^{2z}}{z^2(z^2+2z+2)} + \int \log(z-6) dz + \int \frac{1}{(z-4)^2} dz$$

$$I_1 + I_2 + I_3$$

Using Cauchy Integral formula

$$I_1: \int \frac{e^{2z}}{z^2(z^2+2z+2)} dz = \frac{2\pi i}{2!} \times \left(\frac{e^{2z}}{z^2+2z+2} \right)_{z=0}$$

$$= \pi i \left(\frac{2e^{2z}(z^2+2z+2) - (2z+2)e^{2z}}{(z^2+2z+2)^2} \right)_{z=0}$$

$$= \pi i \left(\frac{4e^{2z}(z^2+2z+2) + 2e^{2z}(2z+2) - 2e^{2z}(2z+2) - 2e^{2z}}{(z^2+2z+2)^2} - 2(2z+2) \frac{2e^{2z}}{(z^2+2z+2)} - (2z+2)e^{2z}}{(z^2+2z+2)^2} \right)$$

$$= \pi i \left(\frac{(4(2) + 2(2) - 2(2) - 2)(4) - 4(4-2)}{4^2} \right)$$

$$= \pi i \left[\frac{2}{1} \right] = 2\pi i$$

$$\pi i \left[\frac{.16 - 8}{4^2} \right] = \frac{\pi i 2^3}{2^4} = \frac{\pi i}{2}$$

$z_2, z_3 = 0$ as no singularity
lie in $|z| = 3$.

Statement

If $f(z)$ is analytic in and on a closed curve C travelled in anticlockwise direction then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

Prove that a function which has no singularity in the finite part of the plane or at infinity is constant. (10)

Function has no singularity in finite part of plane

$\therefore$ it can be expanded by Taylor series about origin

i.e. $f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{--- (1)}$

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \frac{1}{z^n} \quad \text{--- (2)}$$

Given that $f(z)$ has no singularity at infinity.

$\therefore f\left(\frac{1}{z}\right)$ has no singularity at origin

$\therefore$ no principal part in expansion of $f\left(\frac{1}{z}\right)$ about $z=0$

i.e. $f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} b_n z^n \quad \text{--- (3)}$

from (2) and (3)

$$b_n = a_n = 0 \quad \forall \quad n \geq 1$$

$$\Rightarrow \cancel{f\left(\frac{1}{z}\right)} \text{ and } a_0 = b_0$$

put in (1)

$$f(z) = a_0 + \xi_0(z^n)$$

$$f(z) = a_0$$

i.e. $f(z)$ is constant



Q.7 (a)

Let K be a finite field. Show that the numbers of elements in K is p^n , where p is a prime, which is characteristic of K and $n \geq 1$ is an integer. Also, prove that $\frac{\mathbb{Z}_p[X]}{(X^n - 1)}$ is a field. How many elements does this field have? (20)

We know characteristic is order of element when considered as ^{additive} group.

i.e. $\exists a \in K$

such that $o(a) = p$

Let $H = \langle a \rangle \Rightarrow o(H) = p$

$o(H) \mid o(R) \Rightarrow o(R) = q_1^{x_1} (q_2^{x_2}) \dots (q_n^{x_n})$

(G is finite as R is finite)

By Cauchy theorem

$\exists b \in R$ such that $o(b) = q_i$

where q_i is prime.

Now $(p, q_i) = 1$

but $o(b) = p$

(as all elements have

same additive order in

$\Rightarrow p = q_i$

$\Rightarrow o(R) = p^n$

To prove $\frac{\mathbb{Z}_3[x]}{\langle x^2 + 1 \rangle}$ is field

$x^2 + 1$ is irreducible polynomial over $\mathbb{Z}_3[x]$ as

$$\begin{aligned} 0^2 + 1 &\neq 0 \\ 1^2 + 1 &\neq 0 \\ 2^2 + 1 &\neq 0 \end{aligned}$$

① $\mathbb{Z}_3$ is field (as 3 is prime)

① $\Rightarrow \mathbb{Z}_3$ is integral domain

$\Rightarrow \mathbb{Z}_3[x]$ is integral domain.

Also $\mathbb{Z}_3$ is field

② $\Rightarrow$ irreducible polynomial of $\mathbb{Z}_3[x]$ is irreducible element of $\mathbb{Z}_3[x]$

Also we know that An ideal

③ $\langle n_0 \rangle$ is maximal in PID

$\Leftrightarrow n_0$ is irreducible element

④ $\mathbb{Z}_3[x]$ is PID as $\mathbb{Z}_3$ is field.

$\therefore \langle x^2 + 1 \rangle$ is maximal ideal of ID $\mathbb{Z}_3[x]$.

⑤ $\therefore \frac{\mathbb{Z}_3[x]}{\langle x^2 + 1 \rangle}$ is field.

$$\frac{Z_3[x]}{\langle x^2 + 1 \rangle} = \frac{f(x)}{\langle x^2 + 1 \rangle} = \left(\cancel{a_0 + a_1 x + a_2 x^2} \right) + \langle x^2 + 1 \rangle$$

$$f(x) \in Z_3[x]$$

$Z_3[x]$ is $\mathbb{H}\mathbb{D}$ Euclidean domain

$\Rightarrow$ as Z_3 is field
 $\therefore$ Division algorithm valid.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$f(x) = \cancel{(A+Bx)} \frac{g(x)}{g(x)} (x^2+1) + r(x)$$

where $r(x) = ax + b$.

$$\Rightarrow f(x) + \langle x^2 + 1 \rangle = r(x)$$

$$= ax + b$$

$$a, b \in Z_3$$

a has 3 choice (0, 1, 2)

b has 3 choice (0, 1, 2)

$\therefore 3 \times 3 = 9$ elements in

$$\frac{Z_3[x]}{\langle x^2 + 1 \rangle}$$

Q.7 (b) Show that if $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$, then its derivative $S'(x) = \sum_{n=1}^{\infty} \frac{1}{n^2(1 + nx^2)^2} x$. (15)

$$S(x) = \sum_{n=1}^{\infty} f_n(x)$$

$$f_n(x) = \frac{1}{n^3 + n^4 x^2}$$

$$f_n'(x) = \frac{- (n^4 (2x))}{(n^3 + n^4 x^2)^2} = \frac{-2n^4 x}{(n^3 + n^4 x^2)^2}$$

(1) $f_n'(x)$ is continuous (Rational function with $\Delta \neq 0$)

$$f_n''(x) = \frac{-2n^4 \left[(n^3 + n^4 x^2)^2 - 2(n^4 2x)(n^3 + n^4 x^2)n \right]}{(n^3 + n^4 x^2)^4}$$

$$f_n''(x) = 0$$

$$\Rightarrow (n^3 + n^4 x^2) - 4n^4 x^2 = 0$$

$$\Rightarrow x^2 = \frac{n^3}{n^4(3)} = \frac{1}{3n}$$

$$\text{at } x^2 = \frac{1}{3}$$

$$f_n'''(x) = -2n^4 \left[\frac{-2(2n^4 x)}{(n^3 + n^4 x^2)^3} - 4n^4 \left(\frac{2x(n^3 + n^4 x^2)^3 - 3(n^3 + n^4 x^2)^2 (2n^4 x^2)}{(n^3 + n^4 x^2)^6} \right) \right]$$

$$f_n'''(x) \Big|_{x=\frac{1}{\sqrt{3n}}} < 0$$

$\Rightarrow f'_n(x) \Big|_{x=\frac{1}{\sqrt{3n}}}$ is maximum

$$\Rightarrow |f'_n(x)| < \left| f'_n\left(\frac{1}{\sqrt{3n}}\right) \right|$$

$$f'_n(x) < \left| \left(\frac{3 - 2\sqrt{\frac{1}{3n}}}{\left(\frac{4n}{3}\right)^2} \right) \right|$$

$$< \left| \frac{3\sqrt{3}}{8} \cdot \frac{1}{n^{5/2}} \right|$$

$$f'_n(x) < \left| \frac{1}{n^{5/2}} \right|$$

$\frac{1}{n^{5/2}} < \frac{1}{n^{5/2}}$ is convergent

by p test as $5/2 > 1$

$\therefore \sum f'_n(x)$ is convergent

by Weierstrass M-test

(2) $\Rightarrow \sum f'_n(x)$ is uniformly convergent

~~$\Rightarrow$ Term~~

(3) Also $\frac{1}{n^3 + n^4 x^2}$ is differentiable
as denominator
is never 0
 $n > 0, x^2 > 0$
 x

④ $\sum \frac{1}{n^3 + n^4 x^2}$ converges to $s(x)$

from 1, 2, 3, 4

$$\Rightarrow \sum \left(\frac{1}{n^3 + n^4 x^2} \right)' = s'(x)$$

$$\Rightarrow \frac{-2x/n^4}{(n^3 + n^4 x^2)^2} = s'(x)$$

$$s'(x) = -2x \sum \frac{1}{(n + n^2 x^2)^2}$$

(c) Evaluate the residues of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at $z = 1, 2, 3$ and infinity and show that their sum is zero.

(15)

$z = 1, 2, 3$ are simple poles.

Residue at $z = 1$ is $\lim_{z \rightarrow 1} \frac{(z-1) z^3}{(z-1)(z-2)(z-3)}$

$$= \frac{1}{2}$$

Residue at $z = 2$ is $\lim_{z \rightarrow 2} \frac{(z-2) z^3}{(z-1)(z-2)(z-3)}$

$$= \frac{8}{-1} = -8$$

Residue at $z = 3$ is $\lim_{z \rightarrow 3} \frac{(z-3) z^3}{(z-1)(z-2)(z-3)}$

$$= \frac{27}{2}$$

Residue at infinity

$$= -\text{coeff of } \frac{1}{z} \text{ in } f\left(\frac{1}{z}\right)$$

ie $f\left(\frac{1}{z}\right) = \frac{\left(1 - \frac{1}{z}\right)^3 \left(1 - \frac{2}{z}\right)^{-1} \left(1 - \frac{3}{z}\right)^{-1}}{1 + \frac{6}{z} + 0\left(\frac{1}{z^2}\right)}$

$$f\left(\frac{1}{z}\right) = \left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right) \left(1 + \frac{2}{z} + \dots\right) \left(1 + \frac{3}{z} + \dots\right)$$

Residue at ∞ is -6

Sum of residue at 0 is

$$\sum \text{Residue at } 0 + \text{Residue at } \infty = \frac{1}{2} - 8 + \frac{27}{8} = 6$$

$$= 6 + (-6) = 0$$

- 10) If $w_1 = \{(x, y, z) | x + y - z = 0\}$, $w_2 = \{(x, y, z) | 3x + y - 2z = 0\}$,
 $w_3 = \{(x, y, z) | x - 7y + 3z = 0\}$ then
 find $\dim(w_1 \cap w_2 \cap w_3)$ and $\dim(w_1 + w_2)$.

$$\dim(w_1 \cap w_2 \cap w_3) \rightarrow$$

$$x + y - z = 0$$

$$3x + y - 2z = 0$$

$$x - 7y + 3z = 0$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & -2 \\ 1 & -7 & 3 \end{bmatrix}$$

$$\begin{array}{l} R_3 \rightarrow R_3 - R_1 \\ \sim \\ R_2 \rightarrow R_2 - 3R_1 \end{array} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & -8 & 4 \end{bmatrix}$$

$$\begin{array}{l} R_3 \rightarrow R_3 - 4R_2 \\ \sim \end{array} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \dim(\text{solution space}) &= n - \rho(A) \\ &= 3 - 2 = 1 \end{aligned}$$

$$\dim(w_1 \cap w_2 \cap w_3) = 1$$

$$\dim(w_1 + w_2) = \dim(w_1) + \dim(w_2) - \dim(w_1 \cap w_2)$$

$$\dim(w_1) = 2 = \dim(w_2)$$

$$\dim(w_1 \cup w_2) \Rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A \xrightarrow{R_2 \Rightarrow R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

$$\dim(A) = 2$$

$$\Rightarrow \dim(w_1 \cup w_2) = 2$$

$$\dim(w_1 + w_2) = 2 + 2 - 2 = 2.$$

Doubt : How to calculate $\dim(w_1 + w_2)$

without formula $\dim(w_1 + w_2) = \dim(w_1) + \dim(w_2) - \dim(w_1 \cap w_2)$

i.e. by matrix and ~~not so~~