

(To be filled by candidate)

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Exam Centre : Old Rajinder Nagar ☒ Online ☐

CSE (MAINS) TEST SERIES-2024

Test-02

MATHEMATICS

Dated : 30-06-2024

OPTIONAL

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are **EIGHT** questions divided in **Two sections** and printed in **ENGLISH**.

Candidate has to attempt **FIVE** questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

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Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	20			
2. (b)	18	15			
2. (c)	21	15			
3. (a)	24	20			
3. (b)	27	15			
3. (c)	30	15			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	40	20			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	57	15			
6. (c)	61	15			
7. (a)	64	20			
7. (b)	67	15			
7. (c)	70	15			
8. (a)	74	20			
8. (b)	77	15			
8. (c)	81	15			
Grand Total					

Remarks

Observations:

SECTION-A

Q.1 (a) Show that every group of order 15 is cyclic.

(10)

Let G be a group of order 15. Let us assume that G is not cyclic.

Let $a (\neq e)$ be an element of G .

Then $O(G) = 15 \Rightarrow O(a) \leq 15$.

$O(a) \neq 1$ (as $a \neq e$) $\Rightarrow O(a) = 3 \text{ or } 5 \text{ or } 15$
(By Lagrange's theorem)

If $O(a) = 15$ then G is a cyclic group spanned by a .

Case (i) : $O(a) = 3$. Then $\{a, a^2, e\}$ is the cyclic group by a . Here $(a^2)^{-1} = a$.

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- Q.1 (b) Show that the smallest subgroup V of A_4 containing $(1, 2)(3, 4)$, $(1, 3)(2, 4)$ and $(1, 4)(2, 3)$ is isomorphic to Klein 4-group. (10)

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Q.1 (c) If $\sum u_n$ converges, prove that $\sum \frac{u_n}{1-u_n}$, ($u_n \neq 1$) also converges. (10)

Given that $\sum u_n$ converges.

Let $S_n = \sum \frac{u_n}{1-u_n}$. Applying Ratio test for S_n :

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{S_n}{S_{n+1}} &= \lim_{n \rightarrow \infty} \frac{\frac{u_n}{1-u_n}}{\frac{u_{n+1}}{1-u_{n+1}}} \\ &= \lim_{n \rightarrow \infty} \frac{u_n - u_n u_{n+1}}{u_{n+1} - u_n u_{n+1}} \quad \text{--- (1)} \end{aligned}$$

Since $\sum u_n$ is convergent, it obeys ratio test

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = l \quad \text{--- (2) } (l \text{ is finite and } \neq 0)$$

From (1) & (2),

$$\lim_{n \rightarrow \infty} \frac{u_n - u_n u_{n+1}}{u_{n+1} - u_n u_{n+1}} < l$$

$$(\because \frac{a}{b} < \frac{a-l}{b-l})$$

Therefore from (1), $\sum \frac{u_n}{1-u_n}$ also obeys the ratio test.

$\therefore$ By Ratio test, $\sum \frac{U_n}{1-U_n}$ is convergent
Hence proved.

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Q.1 (d) Give the economic interpretation of duality in linear programming.

(10)

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Let us consider the L.P.P

$$\text{Max } Z = a_1x_1 + a_2x_2 + \dots + a_nx_n$$

subject to the constraints

$$b_1x_1 + b_2x_2 + \dots + b_nx_n \leq \alpha$$

$$c_1x_1 + c_2x_2 + \dots + c_nx_n \leq \beta$$

$$d_1x_1 + d_2x_2 + \dots + d_nx_n \leq \gamma$$

Now the dual of the L.P.P is given by

$$\text{Min } Z^* = \alpha x_1^* + \beta x_2^* + \gamma x_3^* \text{ subject to}$$

$$b_1x_1^* + c_1x_2^* + d_1x_3^* \geq a_1$$

$$b_2x_1^* + c_2x_2^* + d_2x_3^* \geq a_2$$

⋮

$$b_nx_1^* + c_nx_2^* + d_nx_3^* \geq a_n$$

The economic significance of the dual is given by the hidden constraints in the new set of equations helping us to solve the L.P.P by methods like simplex etc.

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- Q.1 (e) Obtain the first three terms of Laurent Series expansion of the function $f(z) = \frac{1}{e^z - 1}$ about the point $z = 0$ valid in the region $0 < |z| < 2\pi$. (10)

$$f(z) = \frac{1}{e^z - 1} \quad e^z = 1 + z + \frac{z^2}{2!} + \dots$$

~~$$e^z = 1 + z + \frac{z^2}{2!} + \dots$$~~

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

In the region $|z| < 2\pi$, $\left(\frac{|z|}{2\pi} < 1\right)$

$$\frac{1}{e^z - 1} = \frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots}$$

$$= \frac{1}{z \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots\right)} = \frac{1}{z} \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \dots\right)^{-1}$$

By expansion $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots$

$$\Rightarrow \frac{1}{e^z - 1} = \frac{1}{z} \left(1 - \left(\frac{z}{2} + \frac{z^2}{3!} + \frac{z^3}{4!} + \dots\right) + \left(\frac{z}{2} + \frac{z^2}{3!} + \frac{z^3}{4!} + \dots\right)^2 - \dots\right)$$

$$\Rightarrow \frac{1}{e^z - 1} = \frac{1}{z} \left(1 - \left(\frac{z}{2} + \frac{z^2}{3!} + \frac{z^3}{4!} + \dots\right) + \left(\frac{z^2}{4} + \frac{z^4}{(3!)^2} + \dots\right) - \dots\right)$$

First three terms of the RHS are:

$$\begin{aligned} \Rightarrow \frac{1}{e^z - 1} &= \frac{1}{z} \left(1 - \frac{z}{2} - \frac{z^2}{6} + \frac{z^2}{4}\right) \\ &= \frac{1}{z} \left(1 - \frac{z}{2} + \frac{z^2}{12}\right) = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} \end{aligned}$$

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Hence, the first three terms are as:

$$\frac{1}{e^2 - 1} = \frac{1}{2} - \frac{1}{2} + \frac{2}{12}$$

- Q.2 (a) Let S be the set of all real numbers except -1 . Define $*$ on S by $a * b = a + b + ab$. Is $(S, *)$ a group? Find the solution of the equation $2 * x * 3 = 7$ in S . (20)

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$S = \{R - \{-1\}\}$ (Given). Checking if $(S, *)$ is a group:

(i) Closure property: Let $a, b \in S$ ($a, b \neq -1$)

Then, $a * b = a + b + ab$.

If $a + b + ab = -1$ then $a + ab = -1 - b$

$$\Rightarrow a(b+1) = -1-b \Rightarrow (a+1)(b+1) = 0$$

$$\Rightarrow a+1=0 \text{ or } b+1=0 \text{ i.e. } a=-1 \text{ or } b=-1$$

But $a, b \neq -1$ ($\because a, b \in S$)

Hence $a * b = a + b + ab \neq -1$ — (1)

~~$a * b \in S$~~ $\because a, b \in R, a + b + ab \in R$ — (2)

$\therefore$ From (1) & (2), $a + b + ab \in \{R - \{-1\}\}$

$\Rightarrow \underline{a * b \in S}$ $\therefore$ Closure property satisfied.

(ii) Associative property. Let $a, b, c \in S$.

~~$$\text{Then, } a * (b * c) = a * (b + c + bc)$$~~

~~$$= (a * b) + (a * c) + (a * bc)$$~~

~~$$= (a + b + ab) + (a + c + ac) + (a + bc + abc)$$~~

~~$$= 3a + b + c + ab + ac + abc$$
 — (3)~~

~~Considering $(a * b) * c$:~~

$$(a \times b) \times c = (a + b + ab) \times c = (a \times c) + (b \times c) + (ab \times c)$$

$$\Rightarrow (a \times b) \times c = a + c + ac + b + c + bc$$

$$+ ab + c + abc$$

$$\Rightarrow a + b + 3c + ab + bc + ac + abc$$

Consider $a \times (b \times c)$:

$$a \times (b \times c) = a \times (b + c + bc)$$

$$\Rightarrow a \times (b \times c) = a + (b + c + bc) + a(b + c + bc)$$

$$= a + b + c + bc + ab + ac + abc$$

$\hookrightarrow \textcircled{1}$.

Consider $(a \times b) \times c$:

$$(a \times b) \times c = (a + b + ab) \times c$$

$$\Rightarrow (a \times b) \times c = (a + b + ab) + c + (a + b + ab)c$$

$$\Rightarrow (a \times b) \times c = a + b + ab + c + ac + bc + abc$$

$$= a + b + c + ab + bc + ac + abc$$

$\hookrightarrow \textcircled{2}$.

From $\textcircled{1}$ & $\textcircled{2}$, $a \times (b \times c) = (a \times b) \times c$

Hence Associativity satisfied.

III) Identity property. We know that

$$0 \in \{R - \{-1\}\} \text{ i.e. } 0 \in S.$$

Now let $a \in S$.

$$\text{Then } a \times 0 = a + 0 + (a \times 0) = a$$

$$\text{Also } 0 * a = 0 + a + (0)(a) = a$$

$$\therefore \underline{0 * a = a * 0 = a}$$

$\therefore 0$ is the identity element.

$\Rightarrow$ Identity property satisfied.

(IV) Inverse property: Let $a, b \in S$ & $a * b = 0$

$$a * b = a + b + ab = 0 \text{ (say)}$$

$$\text{Then: } a + ab + b = 0 \Rightarrow a(b+1) + b = 0$$

$$\Rightarrow a = \frac{-b}{b+1} \quad \left(\because b+1 \neq 0 \text{ (as } b \neq -1) \right)$$

$$\text{Hence } \forall b \in S, \exists a = \frac{-b}{b+1} \Rightarrow$$

$$a * b = 0 \text{ i.e. } a \text{ is inverse of } b.$$

Hence Inverse property satisfied.

From (I), (II), (III), (IV), We can clearly see that $(S, *)$ is a group.

$$\text{Given that } 2 * x * 3 = 7.$$

Since $(S, *)$ is a group and obeys associative property,

$$2 * (x * 3) = 7$$

$$\Rightarrow 2 * (x + 3 + 3x) = 7 \quad (\because a * b = a + b + ab)$$

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$$\Rightarrow 2 \times (4x + 3) = 7$$

$$\Rightarrow 2 + (4x + 3) + 2(4x + 3) = 7$$

$$\Rightarrow 2 + 4x + 3 + 8x + 6 = 7$$

$$\Rightarrow 12x + 11 = 7 \Rightarrow 12x = -4$$

$$\Rightarrow \boxed{x = -\frac{1}{3}} \quad (x \in \mathbb{R})$$

is the reqd solution.

- Q.2 (b) If $\langle a_n \rangle$ is a sequence of positive real numbers such that $a_n = \frac{1}{2}(a_{n-1} + a_{n-2}) \forall n \geq 3$, then show that $\langle a_n \rangle$ converges to $\frac{1}{3}(a_1 + 2a_2)$. (15)

To show that $\langle a_n \rangle$ is convergent and $\langle a_n \rangle \rightarrow \frac{1}{3}(a_1 + 2a_2)$.

$$a_3 = \frac{1}{2}(a_1 + a_2) \equiv \cdot \text{let us assume}$$

$$a_2 > a_1 \quad \text{and} \quad \underbrace{a_n > a_{n-1}}_{\hookrightarrow (1)} \quad ; \quad \underbrace{a_{n+1} > a_n}_{\hookrightarrow (2)}$$

Adding the inequalities (1) and (2),

$$a_n + a_{n+1} > a_n + a_{n-1}$$

$$\Rightarrow \frac{1}{2}(a_{n+1} + a_n) > \frac{1}{2}(a_n + a_{n-1})$$

$$\Rightarrow \underline{a_{n+2} > a_{n+1}}$$

$\Rightarrow$ By the induction theorem, $\langle a_n \rangle$ is monotonic.

Lower bound of $\langle a_n \rangle = a_1$.

$$\text{Now } a_n = \frac{1}{2}(a_{n+1} + a_{n-2})$$

$$\Rightarrow a_n = \frac{1}{2} \left(\frac{1}{2}(a_{n-2} + a_{n-3}) + \frac{1}{2}(a_{n-3} + a_{n-4}) \right)$$

$$= \frac{1}{2} \left(\frac{1}{2}(a_{n-2}) + a_{n-3} + \frac{1}{2}(a_{n-4}) \right)$$

$$= \frac{1}{2} \left(\frac{1}{2} \left(\frac{1}{2}(a_{n-3} + a_{n-4}) \right) + a_{n-3} + \frac{1}{2}(a_{n-4} + a_{n-5}) \right)$$

Consider $a_3 = \frac{1}{2}(a_1 + a_2)$. Then, $a_4 = \frac{1}{2}(a_3 + a_2)$

$$a_4 = \frac{1}{2}\left(a_2 + \frac{a_1}{2} + \frac{a_2}{2}\right) = \frac{1}{2}\left(\frac{a_1}{2} + \frac{3a_2}{2}\right)$$

$$\begin{aligned} a_5 &= \frac{1}{2}(a_4 + a_3) = \frac{1}{2}\left(\frac{a_1}{4} + \frac{3a_2}{4} + \frac{a_1}{2} + \frac{a_2}{2}\right) \\ &= \frac{1}{2}\left(\frac{3a_1}{4} + \frac{5a_2}{4}\right) \end{aligned}$$

$$a_6 = \frac{1}{2}(a_5 + a_4) = \frac{1}{2}\left(\frac{3a_1}{8} + \frac{5a_2}{8} + \frac{a_1}{4} + \frac{3a_2}{4}\right)$$

$$\Rightarrow a_6 = \frac{1}{2}\left(\frac{5a_1}{8} + \frac{11a_2}{8}\right)$$

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Q.2 (c) Solve graphically the following LPP:

$$\text{Max } z = 5x_1 - 3x_2$$

subject to

$$3x_1 + 2x_2 \leq 12$$

$$-x_1 + x_2 \geq 1$$

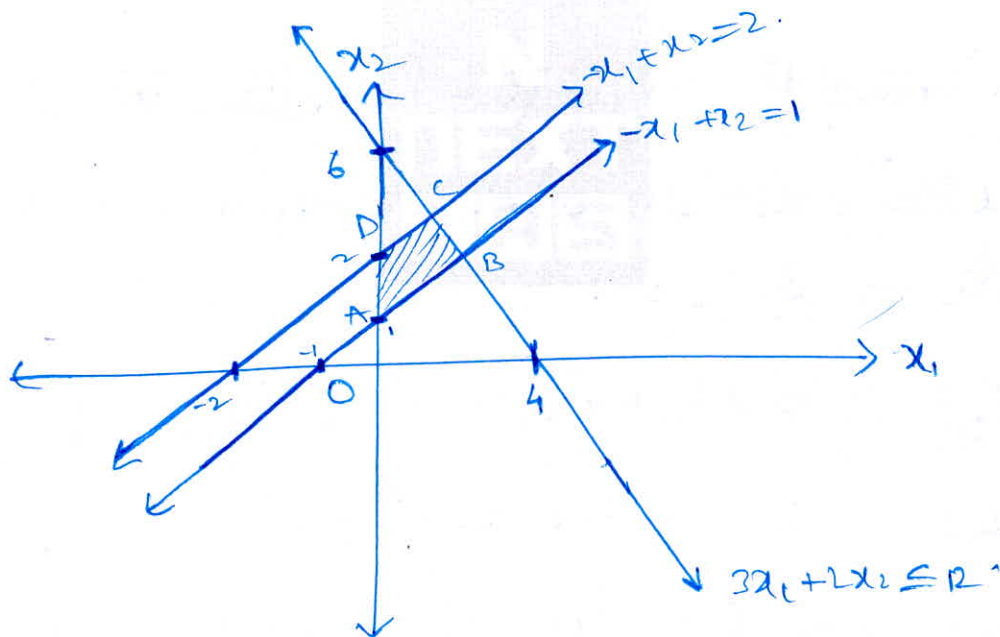
$$-x_1 + x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

If the objective function is changed to $\text{Max } z = 6x_1 + 4x_2$, while the constraints remain the same, then comment on the number of solutions. Will $(4, 0)$ be also a solution? (15)

Given constraints: $3x_1 + 2x_2 \leq 12$;
 $-x_1 + x_2 \geq 1$, $-x_1 + x_2 \leq 2$.

Constructing graph of the constraints:



The shaded area gives the possible region of the solutions.

Corner points A, B, C, D are given by

$$A(0, 1); B(2, 3); C\left(\frac{8}{5}, \frac{18}{5}\right); D(0, 2)$$

Values of z at the reqd points:

At A; $z = 5x_1 - 3x_2 = 5(0) - 3(1) = -3$

At B; $z = 5x_1 - 3x_2 = 5(2) - 3(3) = 1$

At C; $z = 5x_1 - 3x_2 = 5\left(\frac{8}{5}\right) - 3\left(\frac{18}{5}\right) = -\frac{14}{5}$

At D; $z = 5x_1 - 3x_2 = 5(0) - 3(2) = -6$

Hence the solution occurs at $B(2, 3)$

$\therefore \text{Max } z = 1$ at $x_1 = 2$ and $x_2 = 3$

Changing the objective function to

$\text{Max } z = 6x_1 + 4x_2$ and checking z at the points A, B, C, D:

At A; $z = 6x_1 + 4x_2 = 6(0) + 4(1) = 4$

At B; $z = 6x_1 + 4x_2 = 6(2) + 4(3) = 24$

At C; $z = 6x_1 + 4x_2 = 6\left(\frac{8}{5}\right) + 4\left(\frac{18}{5}\right) = 24$

At D; $z = 6x_1 + 4x_2 = 6(0) + 4(2) = 8$

Max z at $B = C = 24$.

Checking if other points on the segment BC (on $3x_1 + 2x_2 = 12$) have the solution z :

Any pt on BC = $\left(k, 6 - \frac{3k}{2}\right)$ which

satisfy $3x_1 + 2x_2 = 12$

$$Z = 6x_1 + 4x_2 = 6(k) + 4\left(6 - \frac{3k}{2}\right) \\ = 6k + 24 - 6k = 24.$$

$\therefore$ Max $Z = 24$ for all points on the segment BC.

$\Rightarrow$ There exist infinitely many solutions to Max $Z = 6x_1 + 4x_2$ in the region.

$(4, 0) = (x_1, x_2)$ is not a solution as it does not obey the constraint $-x_1 + x_2 \geq 1$. $\therefore$ It will not be a solution.
($\because (-4 + 0) \neq 1$)

- Q.3 (a) Let M be a subgroup and N a normal subgroup of a group G . show that MN is a subgroup of G and MN/N is isomorphic to $M/(M \cap N)$.

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Q.3 (b) Every bounded sequence has at least one limit point.

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- Q.3 (c) If all zeros of a polynomial lies in a half plane then show that zeros of derivative $P'(z)$ also lies in the same half plane. (15)

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- Q.4 (a) Let G be a finite commutative group. Let $n \in \mathbb{Z}$ be such that n and order of G are relatively prime. Show that the function $\phi: G \rightarrow G$ defined by $\phi(a) = a^n$ for all $a \in G$, is an isomorphism of G onto G . (15)

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Given that $\phi: G \rightarrow G$ where $\phi(a) = a^n$
 $n, o(G)$ are relatively prime $\Rightarrow \text{GCF} = 1$.

(I) Proving ϕ is homomorphism:

Consider $\phi(a) = a^n$ and $\phi(b) = b^n$
 where $a, b \in G$.

Then $\phi(a) \times \phi(b) = a^n \times b^n = (ab)^n$

$\Rightarrow \phi(a) \cdot \phi(b) = (ab)^n = \phi(ab)$

$\Rightarrow \phi(ab) = \phi(a) \phi(b)$. Hence ϕ is
 an isomorphism.

(II) Proving ϕ is one-one.

Consider $\phi(a) = \phi(b) \Rightarrow a^n = b^n$

$a^n = b^n$. Multiplying $(b^n)^{-1}$ on both sides:

$\Rightarrow a^n (b^n)^{-1} = b^n (b^n)^{-1} = e$

$\Rightarrow a^n b^{-n} = e \Rightarrow (ab^{-1})^n = e$.

Since n and $o(G)$ are relatively
 prime, $(ab^{-1})^n = e$ only if $ab^{-1} = e$

$$\Rightarrow ab^{-1} = e \Rightarrow (ab^{-1}) \cdot b = e \cdot b = b$$

$$\Rightarrow a \cdot (bb^{-1}) = b \Rightarrow a \cdot e = b$$

$$\Rightarrow \underline{a = b}.$$

$$\therefore \phi(a) = \phi(b) \Rightarrow a = b.$$

Hence ϕ is one-one.

III) Proving that ϕ is onto:

Considering $a^n \in G$ where $n, o(G)$ are relatively prime,

$a^n = a \cdot a \cdot a \cdots a$. Since G is a group and obeys closure property, ~~and~~

$$\exists a \in G \Rightarrow \underline{\phi(a) = a^n}.$$

Hence ϕ is onto.

From I, II and III, ϕ is an isomorphism of G onto G .

Hence proved.

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Q.4 (b)

Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify. (15)

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Consider $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$ over

the interval $[0, 1]$.

By the sequence criterion / Heine theorem

$f(x)$ is discontinuous $\forall x \in [0, 1]$

Calculating Riemann Integral for $f(x)$:

Consider n intervals $[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \dots, [\frac{n-1}{n}, \frac{n}{n}]$ of width $\frac{1}{n}$.

Let $\sum m_r \delta_r = L(P, b)$ which

is lower Riemann sum; $\delta_r = \frac{1}{n}$; $m_r = \min_{x \in I_r} f(x)$

$$\Rightarrow L(P, b) = \sum (-1) \delta_r = (-1) \times 1 = -1.$$

Let $\sum M_r \delta_r = U(P, b)$ which is upper

Riemann sum; $\delta_r = \frac{1}{n}$; $M_r = \max_{x \in I_r} f(x)$ in the interval

$$\Rightarrow U(P, b) = \sum (+1) \delta_r = (1) (1) = +1.$$

$$\text{Now } \int_0^1 f(x) dx = \text{Lower Integral} = \sup \{L(P, b)\} = -1 \quad \text{--- (1)}$$

$$\int_0^1 f(x) dx = \text{Upper Integral} = \inf \{U(P, b)\} = +1 \quad \text{--- (2)}$$

From ① and ②, $\int_a^b f dx \neq \int_a^b f dx$

Hence f is not Riemann integrable.

(Also a function is not Riemann integrable if it is discontinuous at infinite points without a limit)

Consider $|f(x)|$ in $[0, 1]$:

$$|f(x)| = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

$$\text{Now } \int_0^1 f(x) dx = \int_0^1 1 dx = 1$$

(Since f is a constant function, it is Riemann integrable)

Hence, f is a function that is not Riemann integrable but $|f(x)|$ is Riemann integrable.

Hence proved.

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Q.4 (c)

A police department has the following minimal daily requirements for police officers during its six shift periods:?

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Time of Day	Period	Minimal number required
2-6 am	1	22
6-10 am	2	55
10 am-2pm	3	88
2-6 pm	4	110
6-10 pm	5	44
10 pm-2 am	6	33

An officer must start at the beginning of a 4 hour shift and stay on duty for two consecutive shifts. Any one starting during period 6 stays on duty during period 1 of the next day. The objective of the police department is to always have on duty the minimal number required in a period but to do so with the least number of officers. Develop the corresponding linear programming problem. (20)

Considering $x_1, x_2, x_3, x_4, x_5, x_6$ be the number of officers who enter duty at the start of periods 1, 2, 3, 4, 5, 6 respectively.

Then, the total number of officers in duty during the period x_p is given by

$n = x_p + x_{p-1}$ i.e. the sum of officers who enter duty during the start of period p and the officers who remain in duty for the consecutive period, starting from $(p-1)^{\text{th}}$ period.

Objective: $\text{Min } Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$.

Constraints:

① Period 1 : min = 22

$$\Rightarrow x_1 + x_6 \geq 22$$

② Period 2 : min = 55

$$\Rightarrow x_2 + x_1 \geq 55$$

③ Period 3 : min = 88

$$\Rightarrow x_3 + x_2 \geq 88$$

④ Period 4 : min = 110

$$\Rightarrow x_4 + x_3 \geq 110$$

⑤ Period 5 : min = 44

$$\Rightarrow x_5 + x_4 \geq 44$$

⑥ Period 6 : min = 33

$$\Rightarrow x_6 + x_5 \geq 33.$$

Hence the L.P.P is given by =

$$\text{Min } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

$$\text{Subject to: } x_1 + x_6 \geq 22;$$

$$x_1 + x_2 \geq 55;$$

$$x_2 + x_3 \geq 88;$$

$$x_3 + x_4 \geq 110;$$

$$x_4 + x_5 \geq 44;$$

$$x_5 + x_6 \geq 33;$$

is the required L.P.P

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SECTION-B

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- Q.5 (a) Let R be a non-commutative ring with unity. If every ideal of R is prime, prove that R is a field. (10)

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- Q.5 (b) Prove that if $u_n(x)$, $n = 1, 2, \dots$ are continuous in $[a, b]$ and if $\sum u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, $S(x)$ is continuous in $[a, b]$. (10)

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Given: $U_n(x)$ is continuous in $[a, b]$

$\sum U_n(x)$ converges uniformly to $S(x)$ in $[a, b]$.

Then $\exists$ a unique $m \in \mathbb{Z}^+$

$$\Rightarrow |\sum U_n(x) - S(x)| < \varepsilon \quad \forall n \geq m \quad \text{--- (1)}$$

Let l be the limit of $S(x)$ where

$$l = \sum U_n(x) \text{ for any } n \geq m.$$

Then choosing any $\varepsilon > 0$,

$$\exists \delta > 0 \Rightarrow |S(x) - l| < \varepsilon \quad \forall x \in [a, b]$$

Hence, $S(x)$ has limit existed $\forall x \in [a, b]$

$\therefore S(x)$ is continuous on $[a, b]$.

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Q.5 (c) Prove that $x_1 = 2, x_2 = 1, x_3 = 0$ is feasible solution to the following set of equations:

$$2x_1 - x_2 + 3x_3 = 3$$

$$-6x_1 + 3x_2 + 7x_3 = -9$$

Is the solution basic? Justify your answer. If the solution is not basic, reduce it to a basic feasible one. (10)

Substituting the values $(2, 1, 0)$ in the equations to check feasibility:

$$2x_1 - x_2 + 3x_3 = 2(2) - (1) + 3(0) = 3$$

and

$$-6x_1 + 3x_2 + 7x_3 = -6(2) + 3(1) + 7(0) = -9$$

Hence the set $(x_1, x_2, x_3) = (2, 1, 0)$ satisfy the equations. $\therefore$ It is a feasible solution.

Taking x_1, x_2 as the basic variables and x_3 as the non basic variable ($x_3 = 0$).

We find that (x_1, x_2, x_3) is a basic solution.

(Basic solution has non zero values for the basic variables and zero values for the ~~non zero~~ non basic variables)

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- Q.5 (d) Evaluate $\int_C \left[\frac{e^{2z}}{z^2(z^2+2z+2)} + \log(z-6) + \frac{1}{(z-4)^2} \right] dz$ where C is the circle $|z|=3$. State the theorem you use in evaluating the above theorem. (10)

Solving the theorem by Cauchy's Integral theorem: If $f(z)$ is not analytic inside a ~~circle~~ region C at $z=a$ then the integral
$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a).$$

Now $\int_C \frac{e^{2z}}{z^2(z^2+2z+2)} + \log(z-6) + \frac{1}{(z-4)^2} dz$

(I) $\int_C \log(z-6) dz = 0$ as the function is fully analytic inside the region $|z|=3$.

(II) $\int_C \frac{1}{(z-4)^2} dz = 0$ also; as the function exists and is analytic for all values inside $|z|=3$

(III) $\int_C \frac{e^{2z}}{z^2(z^2+2z+2)} dz$. The function is non analytic at $z=0$; $z=-1+i$; $z=-1-i$
Let $f(z) = \frac{e^{2z}}{z^2(z^2+2z+2)}$

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$$\oint_C \frac{e^{2z}}{(z-0)^2(z-(1-i))(z-(1+i))} dz.$$

Using the Cauchy's Residue Theorem :

$f(z) = 2\pi i \sum R$ where the residue at pole of order n is $\lim_{z \rightarrow a} \frac{d^{n-1}}{dz^{n-1}} \left((z-a)^n f(z) \right) \times \frac{1}{(n-1)!}$

(i) At $z=0$: Residue is

$$R_1 = \lim_{z \rightarrow 0} \frac{d}{dz} \left(\frac{e^{2z}}{z^2(z^2+2z+2)} \right) = \lim_{z \rightarrow 0} \frac{(z^2+2z+2)(2e^{2z}) - e^{2z}(2z+2)}{(z^2+2z+2)^2}$$

$$\Rightarrow R_1 = \frac{4 - 2}{4} = 1/2.$$

(ii) At $z=-1+i$: Residue is

$$R_2 = \lim_{z \rightarrow -1+i} \frac{e^{2z}}{z^2(z-(1+i))} = \frac{e^{-2+2i}}{(-1+i)^2(-2)} = \frac{e^{-2} e^{2i}}{4i}$$

(iii) At $z=-1-i$, Residue is

$$R_3 = \lim_{z \rightarrow -1-i} \frac{e^{2z}}{z^2(z-(1+i))} = \frac{e^{-2-2i}}{(2i)(-2)} = \frac{e^{-2} e^{-2i}}{-4i}$$

$$\Rightarrow \int_C f(z) dz = 2\pi i \left(\frac{1}{2} + \frac{e^{-2} e^{2i}}{4i} - \frac{e^{-2} e^{-2i}}{4i} \right)$$

$$= 2\pi i \left(\frac{1}{2} + \frac{e^{-2}}{4i} (2i \sin 2) \right)$$

$$= 2\pi i \left(\frac{1}{2} + \frac{e^{-2}}{2} \sin 2 \right) \text{ is the reqd value.}$$

Q.5 (e)

Prove that a function which has no singularity in the finite part of the plane or at infinity is constant. (10)

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Let $f(z)$ be the function, expanding it about the point $z=a$ in Taylor series:

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \dots \quad (1)$$

Since $f(z)$ does not have singularity at $z \rightarrow \infty$;

From (1), $f'(a) = f''(a) = \dots = 0$

Hence $f'(z) = 0 \Rightarrow \underline{z = \text{constant}}$

~~Case (ii):~~

Expanding $f(z)$ about $(z=a)$ in the Laurent series,

$$f(z) = \dots + b_2 \frac{1}{(z-a)^2} + b_1 \frac{1}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$$

Since f does not have singularity at the

finite part, then $b_1 = b_2 = \dots = b_n = 0$.

Hence, clearly $f(z) = \text{constant}$.

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Q.6 (a) Prove that the integral domain Z is UFD and a ED.

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Q.6 (b) Show that $\int_0^{\infty} \frac{dx}{1+x^2 \sin^2 x}$ is divergent.

(15)

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Q.6 (c)

Starting with least cost method, find all the solutions to the following transportation problem:

Warehouse

		I	II	III	IV	Supply
Plants	A	8	6	5	3	18
	B	6	7	6	8	20
	C	10	8	4	5	18
	Demand	15	16	12	13	

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(15)

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- Q.7 (a) If S Let K be a finite field. Show that the numbers of elements in K is p^n , where p is a prime, which is characteristic of K and $n \geq 1$ is an integer. Also, prove that $\frac{Z_3[X]}{(X^2+1)}$ is a field. How many elements does this field have? (20)

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Q.7 (b) Show that if $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$, then its derivative $S'(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 (1 + nx^2)^2} \forall x$. (15)

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- Q.7 (c) Evaluate the residues of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at $z = 1, 2, 3$ and infinity and show that their sum is zero. (15)



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- Q.8 (a) Let V Show that $Q(\sqrt{3}, i)$ is the splitting field for $x^5 - 3x^3 + x^2 - 3$ where Q is the field of rational numbers. (20)

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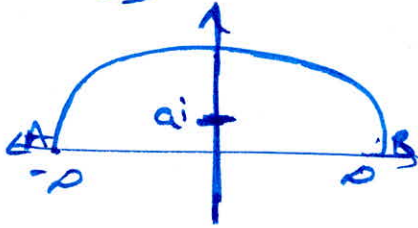
Q.8 (b) Using contour integral method, prove that $\int_0^{\infty} \frac{x \sin mx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ma}$. (15)

Since $\frac{x \sin(mx)}{a^2 + x^2}$ is an even function
(i.e. $f(x) = f(-x)$), therefore $\int_0^{\infty} \frac{x \sin mx}{a^2 + x^2} dx$
 $= \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin(mx)}{a^2 + x^2} dx$.

Consider the Integral $I = \int_{-\infty}^{\infty} \frac{x e^{imx}}{a^2 + x^2} dx$

where $\text{Im}(I) = \int_{-\infty}^{\infty} \frac{x \sin mx}{a^2 + x^2} dx$.

Taking the contour to be an infinite semicircle above the real axis,

$$I = \int_C \frac{x e^{imx}}{x^2 + a^2} dx.$$


$$= \int_{+B} \frac{x e^{imx}}{x^2 + a^2} dx + \int_{BA} \frac{x e^{imx}}{x^2 + a^2} dx.$$

$$\int_{BA} \frac{x e^{imx}}{x^2 + a^2} dx = 0 \quad \text{as} \quad \lim_{x \rightarrow \infty} \frac{x e^{imx}}{x^2 + a^2} = 0$$

$$= \lim_{x \rightarrow 0} \frac{(\cos mx + i \sin mx)}{x + a^2/x} = \underline{\underline{0}}.$$

$$\therefore I = \int_{AB} \frac{ze^{imx}}{x^2+a^2} dx = \oint_C \frac{ze^{imx}}{x^2+a^2} dx.$$

Inside the contour, the ~~singularity~~^{pole} lies at $x = ai$ ~~and~~.

$$\Rightarrow \int_C \frac{ze^{imx}}{x^2+a^2} dx = \int_C \frac{ze^{imx}}{(x+ia)(x-ia)} dx.$$

By the ~~the~~ Cauchy's Residue theorem,

$$\oint_C f(z) dz = 2\pi i \sum R^+ \text{ where } R \text{ is the residues at } \text{the poles}.$$

$$\text{and } R = \lim_{z \rightarrow a} (z-a)f(z).$$

Hence Residue at the pole $z = ai$ is

$$\begin{aligned} \lim_{z \rightarrow ai} (z-ai) \frac{ze^{imx}}{(x+ia)(x-ia)} &= \frac{(ia) \times e^{im(ia)}}{2ia} \\ &= \frac{1}{2} e^{-ma}. \end{aligned}$$

$$\Rightarrow \int f(z) dz = 2\pi i \left(\frac{1}{2} e^{-ma} \right) = \pi i e^{-ma}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{xe^{imx}}{x^2+a^2} dx = \pi i e^{-ma}.$$

$$\text{Now } \int_{-\infty}^{\infty} \frac{x \sin mx}{x^2+a^2} dx = \text{Im}(\pi i e^{-ma}) = \pi e^{-ma}$$

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$$\begin{aligned}\therefore \int_0^{\infty} \frac{x \sin mx}{x^2 + a^2} dx &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin mx}{x^2 + a^2} dx \\ &= \frac{1}{2} \times \pi e^{-ma} = \frac{\pi}{2} e^{-ma}.\end{aligned}$$

$$\therefore \int_0^{\infty} \frac{x \sin(mx) dx}{x^2 + a^2} = \frac{\pi}{2} e^{-ma}$$

Hence proved.

- Q.8 (c) Apply Rouché's theorem to determine the number of roots of the equation $z^8 - 4z^5 + z^2 - 1 = 0$ that lie inside the circle $|z|=1$. (15)

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Rouché's Theorem: If $f(z)$, $g(z)$ are two analytic functions in the region

R , ~~then~~ $|f(z)| < |g(z)| \quad \forall z \in R$.

Then $(f(z) + g(z))$ and $g(z)$ have the same number of zeros inside R .

Let $g(z) = -4z^5$ and $f(z) = z^8 + z^2 - 1$

$$|g(z)| = |-4z^5| = 4|z^5|$$

$$|f(z)| = |z^8 + z^2 - 1| \leq |z^8| + |z^2| + |-1| \leq 3$$

Hence $|g(z)| > |f(z)|$. By Rouché's theorem.

$g(z) = -4z^5$ have five zeros inside $|z|=1$.

$$\therefore f(z) + g(z) = \underline{z^8 - 4z^5 + z^2 - 1 = 0}$$

also have 5 roots that lie inside $|z|=1$.

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