

NEXT IAS

(To be filled by candidate)

Name of Candidate: Kamlesh Aggarwal

Roll No.: MT 24MATL1011 Date of Examination: 31/7

Mobile Number: 1 mail: milan

Exam Centre: Old Rajinder Nagar ☐ Online ☒

CSE (MAINS) TEST SERIES-2024

Test <u>007</u> <u>12</u>	MATHEMATICS
Dated: 14-07-2024	OPTIONAL

Time Allowed: Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are EIGHT questions divided in Two sections and printed in ENGLISH.

Candidate has to attempt FIVE questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.

The number of marks carried by a question part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question cum Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

DELHI CENTRE:

Old Rajinder Nagar: 27-D, Pusa Road, Mohd. Waris, 110, New Karol Bagh, Mohd. New Delhi-110001 | Phone: 8001300000

Munirpur Nagar: 1402, Main Market, Munirpur Nagar, New Delhi-110001 | Phone: 8001300000

BHOPAL CENTRE: Plot No. 40/2000-2, MP Nagar, Bhopal-462011 | Phone: 0672040112, 8001300000

JAIPUR CENTRE: Plot No. 8 & 7, 3rd Floor, Sree Ganga Nagar, Gandhinagar, Jaipur-302015 | Phone: 01412200111

PRAYAGRAJ CENTRE: 81/1, Sarda Road, Main, Civil Lines, Prayagraj, Uttar Pradesh-211001 | Phone: 800007707

(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	15			
2. (c)	20	20			
3. (a)	24	20			
3. (b)	28	15			
3. (c)	31	15			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	40	20			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	20			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	80	20			
Grand Total					

Remarks

Observe all answers

SECTION-A

1. (a)

Show that a group G satisfying $a^2 = e \forall a \in G$ must be abelian.

(10)

$$a, b \in G$$

$$a^2 = e \Rightarrow a = a^{-1}$$

$$b^2 = e \Rightarrow b = b^{-1}$$

 $\otimes$

$$(ab)^2 = e$$

$$(ab)(ab) = e$$

$$(ab) = (ab)^{-1} = b^{-1}a^{-1}$$

$$ab = ba$$

 $\Rightarrow G$ is abelian

If the subsequences $\langle a_{2n-1} \rangle$ and $\langle a_{2n} \rangle$ of a sequence $\langle a_n \rangle$ converges to the same limit l , the sequence $\langle a_n \rangle$ converges to l . (10)

$$|a_{2n-1} - l| < \epsilon \quad \forall n \geq m_1$$

$$|a_{2n} - l| < \epsilon \quad \forall n \geq m_2$$

$$\text{let } m_3 = \max \{m_1, m_2\}$$

~~Q2~~ To prove $|a_n - l| < \epsilon$
 $\forall n \geq m_3$

Show that the $\sin x$ is uniformly continuous on $[0, \infty)$.

(10)

let $x_1, x_2 \in [0, \infty)$

$$|\sin x_1 - \sin x_2| = 2 \left| \sin \frac{x_1 - x_2}{2} \right| \left| \cos \left(\frac{x_1 + x_2}{2} \right) \right|$$

$$\leq 2 \left| \sin \frac{x_1 - x_2}{2} \right|$$

$$\text{as } \left| \cos \left(\frac{x_1 + x_2}{2} \right) \right| \leq 1$$

~~as~~

$$\leq 2 \left(\frac{x_1 - x_2}{2} \right) \quad \left(\begin{array}{l} \text{as } \sin x \leq x \\ \text{Jordan} \\ \text{inequality} \end{array} \right)$$

$$\leq x_1 - x_2$$

Choose $\delta = \epsilon$

$$|x_1 - x_2| < \delta$$

$$\Rightarrow |\sin x_1 - \sin x_2| < \epsilon.$$

1. (d)

Let $f(z)$ be entire function satisfying $|f(z)| \leq k|z|^2$ for some positive constant k and all z . Show that $f(z) = az^2$ for some constant a . (10)

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-a)^{n+1}} dz$$

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi i} \int \frac{|f(z)|}{|z-a|^{n+1}} |dz|$$

$$\leq \frac{n!}{2\pi i} \int \frac{k|z|^2}{|z-a|^{n+1}} |dz|$$

$$z-a = re^{i\theta}$$

$$\leq \frac{n!}{2\pi i} \int_C \frac{k|a+re^{i\theta}|^2}{e^{i(n+1)\theta} r^{n+1} e^{i(n+1)\theta}} (dz)$$

$$\leq \frac{n!k}{2\pi i} \int_C \frac{(a+r\cos\theta)^2 + (r\sin\theta)^2}{r^{n+1} e^{i(n+1)\theta}} (dz)$$

$$\leq \frac{n!k}{2\pi i} \int_C \frac{(a+r)^2 + r^2}{r^{n+1} e^{i(n+1)\theta}} \frac{d\theta}{i} (i r e^{i\theta})$$

$$\leq \frac{n!k}{2\pi i} \frac{(a+r)^2 + r^2}{r^{n+1}} \int_C \frac{d\theta}{e^{i(n+1)\theta}}$$

$$\leq \frac{n!k}{2\pi i} \frac{(a+r)^2 + r^2}{r^{n+1}} i \int_C \frac{d\theta}{e^{i(n+1)\theta}}$$

Candidates
must not write
on this margin

$$= 0 \quad n \neq 0$$

$$= 2\pi \quad n = 0$$

$$\leq \frac{n!}{z^n} \frac{k \left(\left(\frac{a+z}{z} \right)^2 + 1 \right)}{2\pi}$$

$$|f^n(a)| \leq \frac{n!}{z^n} k \left| \left(\frac{a+z}{z} \right)^2 + 1 \right|$$

$$|f^n(a)| \leq \frac{k \left(\left(1 + \frac{a}{z} \right)^2 + 1 \right)}{z^{n-2}}$$

$$|f^n(a)| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{if } n-2 > 0$$

$$\text{i.e. } |f^n(a)| = 0 \quad \forall n > 2$$

$$\Rightarrow f(z) = a_0 + a_1(z-a) + a_2(z-a)^2$$

now $a, a_0, a_1, a_2 \in \mathbb{C}$

and are arbitrary
~~for a_0~~

$$\text{for } a_0 = a_1 = 0 = a_2$$

$$f(z) = a_2 z^2$$

1. (c)

Show that if the optimal value of an objective function is attained at more than one extreme points then every convex combination of these extreme points also gives the optimal value of the objective function. (10)

Let x, y be two optimal values.
 $\Rightarrow Ax = Ay = M$
 where A is objective function matrix
~~any point~~ convex combination
 of x, y is

$$\lambda x + (1-\lambda)y \quad \lambda \in (0,1)$$

$$A[\lambda x + (1-\lambda)y]$$

$$= \lambda(Ax) + (1-\lambda)Ay$$

$$= \lambda(M) + (1-\lambda)M$$

$$= M$$

$\Rightarrow \lambda x + (1-\lambda)y$ is also optimal value.

2. (a)

Show that the polynomial $25x^4 + 9x^3 + 3x + 3$ is irreducible over the field of rational numbers. (15)

Using Eisenstein's criteria of irreducibility.

$$f(x) = 25x^4 + 9x^3 + 3x + 3.$$

$$\text{let } p = 3 \quad ; \quad p|a_0; p|a_1; p|a_{n-1}; p \nmid a_n$$

$$3|3; \quad 3|3; \quad 3|9; \quad 3 \nmid 25$$

$$\text{also } 3^2 \nmid 3$$

$\Rightarrow f(x)$ is irreducible over the field of rational numbers.

Prove that a bounded function f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there exist a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$. (15)

Candidates must not write on this margin

I. Given $\rightarrow f$ is integrable on $[a, b]$

To prove $\rightarrow \exists P[a, b]$ such that
 $U(P, f) - L(P, f) < \epsilon$

As $f \in R[a, b]$

$$\int_a^b f(x) dx = \int_a^b f(x) dx = \int_a^b f(x) dx.$$

$$\int_a^b f(x) dx = \sup \{ L(P, f) \} < \int_a^b L(P', f) + \frac{\epsilon}{2} \quad \text{--- ii'}$$

Riemann theorem $\rightarrow$ All partitions with norm $< \delta$ ~~(given)~~ ^{and given ϵ} satisfy.

$$U(P, f) < \int_a^b f + \epsilon/2$$

$$L(P, f) > \int_a^b f - \epsilon/2$$

Now using Darboux a partition P' exist.

Also

$$U(P', f) < \int_a^b f + \epsilon/2 \quad \text{--- ii'}$$

i, ii, iii

$$U(P', f) - L(P', f) < \epsilon.$$

Given $U(P, f) - L(P, f) < \epsilon$

To prove $f \in R[a, b]$.

$$U(P, f) \geq \int_a^b f dx$$

$$L(P, f) \leq \int_a^b f dx \leq \int_a^b f dx \leq U(P, f)$$

$$\Rightarrow \int_a^b f dx - \int_a^b f dx \leq U(P, f) - L(P, f) < \epsilon$$

$$\Rightarrow \int_a^b f dx - \int_a^b f dx < \epsilon$$

ϵ can be arbitrarily close to 0

$$\Rightarrow \int_a^b f dx = \int_a^b f dx$$

$$\Rightarrow f \in R[a, b]$$

2. (c)

Consider the following LPP,

Maximise: $2x_1 + 4x_2 + 4x_3 - 3x_4$

Subject to $x_1 + x_2 + x_3 = 4$
 $x_1 + 4x_2 + x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

Use the dual problem to verify that the basic solution (x_1, x_2) is not optimal.

(20)

Max Z: $2x_1 + 4x_2 + 4x_3 - 3x_4 - Ma_1 - Ma_2$

s.t.

$$x_1 + x_2 + x_3 + a_1 = 4$$

$$x_1 + 4x_2 + 0x_3 + x_4 + 0a_1 + a_2 = 8$$

$$x_1, x_2, x_3, x_4, a_1, a_2 \geq 0$$

Solving using big M method.

IBFS $\rightarrow x_1 = x_2 = x_3 = x_4 = 0$
 $a_1 = 4, a_2 = 8.$

C_B	x_B	2	4	4	-3	-M	-M		
		x_1	x_2	x_3	x_4	a_1	a_2	b	θ
-M	a_1	1	1	0	0	1	0	4	4
-M	a_2	1	4	0	1	0	1	8	2
Z_j		-2M	-5M	-4M	-3M	-M	-M	-12M	
$C_j - Z_j$		2+2M	4+5M	4+8M	-3+M	0	0	12M	
			$\uparrow$						
4	x_2	1/2	1	3/2	0	1/2	0	2	
-M	a_2	-1	0	-6	1	-2	1	0	0
Z_j		2+M	4	6+6M	-M	2+2M	-M	8	
$C_j - Z_j$		-M	0	-2-6M	3+M	-3M-2	0	-8	

Candidates must not write in this margin

C_B	x_B	2 x_1	4 x_2	4 x_3	-3 x_4	-M a_1	-M a_2	b	θ
4	x_2	$1/2$	1	$3/2$	0	$1/2$	0	2	2
-3	x_4	-1	0	-6	1	-2	1	0	
C_j		5	4	24	-3	0	-3	0	
0		-3	0	-20	0	-M-3	-M-3		

All $C_j \leq 0$
Optimality reached

$$x_2 = 2 ; x_1 = x_3 = x_4$$

C_B	x_B	x_1	x_2	x_3	x_4	a_1	a_2	b	θ
-M	a_1	$3/4$	0	1	$-1/4$	1	$-1/4$	2	2
4	x_2	$1/4$	1	0	$1/4$	0	$1/4$	2	
2		$-3M+1$	4	-M	$\frac{M}{4}+1$	-M	$\frac{M}{4}+1$	2M	
0		$\frac{3M}{4}+1$	0	M+4	$-\frac{M}{4}-1$	0	$-\frac{5M}{4}$		
0	x_3	$3/4$	0	1	$-1/4$	1	$-1/4$	2	
4	x_2	$1/4$	1	0	$1/4$	0	$1/4$	2	
2		4	4	4	0	4	0	16	
0		-2	0	0	-3	-M-4	-M		
	All $C_j \leq 0$	Optimality reached							

$$x_3 = x_2 = 2$$

$$\text{Max } Z = 16$$

Dual

$$\text{Min } Z: 4y_1 + 8y_2$$

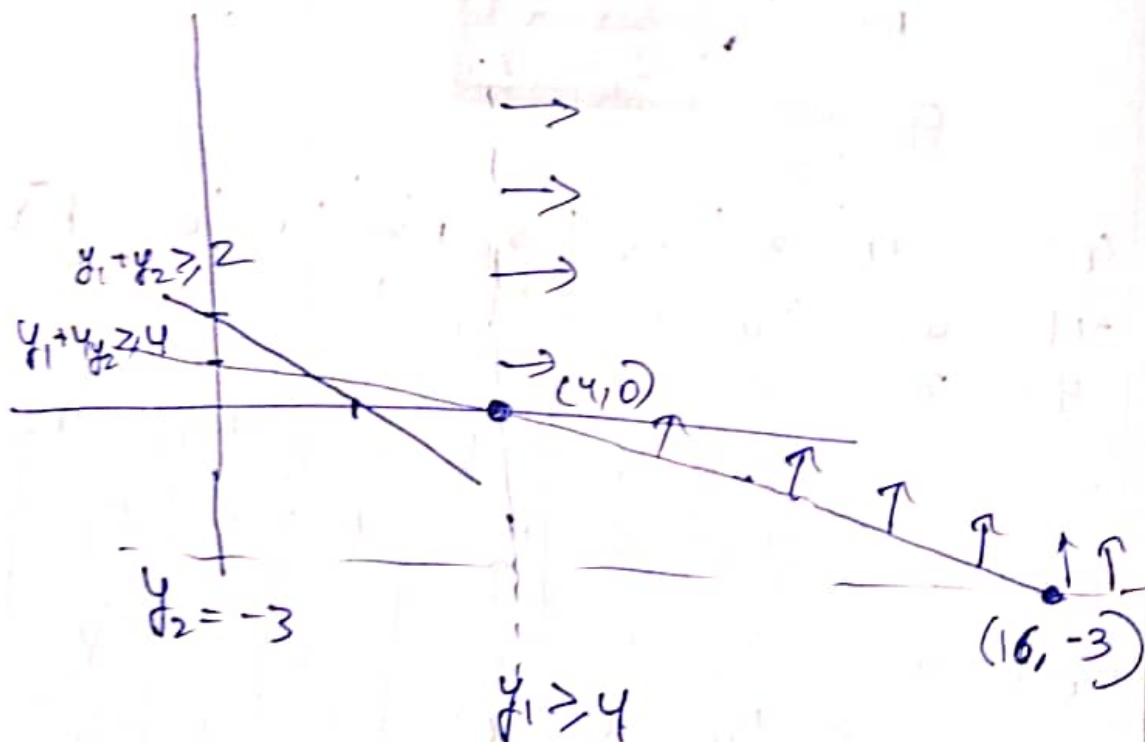
s.c.

$$y_1 + y_2 \geq 2$$

$$y_1 + 4y_2 \geq 4$$

$$y_1 \geq 4$$

$$y_2 \geq -3$$

 y_1, y_2 ~~unrestricted~~
 unrestricted


$$Z(4, 0) = 16$$

$$Z(16, -3) = 40$$

 $(4, 0)$ is optimal

(x_1, x_2) correspond to excess
 variable in $y_1 + y_2 \geq 2$; $y_1 + 4y_2 \geq 4$
 constraint respectively $y_1 + y_2 - e_1 = 2$
 $y_1 + 4y_2 - e_2 = 4$
 since ~~the~~ at $(4, 0)$ $e_1 = +2$
 $e_2 = 0$.

$(e_1, e_2) = (+2, 0)$ is not
 optimal as $\nabla (x_1, x_2)$
 will be $(-2, 0)$ but
 $x_1, x_2 \geq 0$.

3. (a)

Let N be a normal subgroup of a group G . Show that G/N is abelian if and only if for all $x, y \in G$, $xyx^{-1}y^{-1} \in N$. [20]

I

Given $\frac{G}{N}$ is abelian

To prove $xyx^{-1}y^{-1} \in N$

$\frac{G}{N}$ is abelian

let $x, y \in G$

$Nx, Ny \in G/N$

$$Nx \cdot Ny = Ny \cdot Nx$$

$$Nxy = Nyx \quad (\because N \trianglelefteq G)$$

$$(xy)(yx)^{-1} \in N \quad (Ha = Hb \Rightarrow ab^{-1} \in H)$$

$$\Rightarrow xyx^{-1}y^{-1} \in N \quad \square$$

II $\Rightarrow xyx^{-1}y^{-1} \in N \quad \forall x, y \in G$

To prove $\frac{G}{N}$ is abelian

let $x, y \in \frac{G}{N}$

$$xyx^{-1}y^{-1} \in N \quad \forall x, y \in G$$

$$\Rightarrow N(xy x^{-1} y^{-1}) = N$$

$$N(xy (yx)^{-1}) = N$$

$$N(xy) = N(yx) \quad (\because N \trianglelefteq G)$$

$$N(x)N(y) = N(y)N(x) \quad (\because N \trianglelefteq G)$$

i.e. $\frac{G}{N}$ is abelian

$$\therefore Nx, Ny \in G/N$$

and $Nx Ny = Ny Nx.$

3. (b)

Let $\lim_{n \rightarrow \infty} a_n = l$, then prove that $\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + a_3 + \dots + a_n}{n} = l$.

(15)

$$L = \lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{n}$$

$$\text{let } a_i = b_i + l$$

$$L = \lim_{n \rightarrow \infty} \frac{b_1 + b_2 + \dots + b_n}{n} + \frac{n \cdot l}{n}$$

This reduces to proving

$$\lim_{n \rightarrow \infty} \frac{b_1 + b_2 + \dots + b_n}{n} = 0$$

$$\text{Now, } a_i = b_i + l$$

$$\lim_{n \rightarrow \infty} a_n = l$$

$$|a_n - l| < \epsilon \quad \forall n \geq m$$

$$|b_n| < \epsilon \quad \forall n \geq m$$

$$-\epsilon < b_n < \epsilon \quad \forall n \geq m$$

Using this in i

$$\left| \frac{b_1 + b_2 + \dots + b_m}{m} + \frac{b_{m+1} + \dots + b_n}{n} \right|$$

$$\left| \frac{b_1 + \dots + b_m}{m} \right| + \left| \frac{(n-m)\epsilon}{n} \right|$$

Also $|b_n| < \varepsilon \quad \forall n > m$

i.e. $\lim_{n \rightarrow \infty} b_n = 0$

$\Rightarrow b_n$ is bounded

as every convergent sequence is bounded

let $b_n < K \quad \forall n \in \mathbb{N}$

$$|A_n| < \left| \frac{K(m+1)}{n} \right| + \left| \frac{(n-m)\varepsilon}{n} \right| \quad \forall n > m$$

$$|A_n| < \left| \frac{K(m+1)}{n} \right| + |\varepsilon| + \left| \frac{m\varepsilon}{n} \right|$$

$$\lim_{n \rightarrow \infty} |A_n| < \varepsilon \quad \left(\begin{array}{l} \text{as } \lim_{n \rightarrow \infty} \frac{K(m+1)}{n} = 0 \\ \lim_{n \rightarrow \infty} \frac{m\varepsilon}{n} = 0 \end{array} \right)$$

ε can be arbitrarily close to 0

i.e. $|A_n - 0| < \varepsilon \quad \forall n > m$

$\Rightarrow A_n$ converges to 0

i.e.

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n b_i}{n} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n a_i}{n} = L.$$

state Cauchy's Residue Theorem. Using it to evaluate the integral

$$\int_C \frac{e^z + 1}{z(z+1)(z-i)^2} dz, \quad C: |z|=2.$$

(15)

Candidates
must not write
on this margin

Statement: C is closed contour,
 $\frac{1}{2\pi i} \int_C f(z) dz$ is equal to sum of
 residues of $f(z)$ inside C .

Evaluation

$$f(z) = \frac{e^z + 1}{z(z+1)(z-i)^2}$$

Poles $z = 0$ (simple), -1 (simple), i (double)

Residue at $z = 0$ let R_1

$$R_1 = \lim_{z \rightarrow 0} z f(z) = \lim_{z \rightarrow 0} \frac{e^z + 1}{(z+1)(z-i)^2} = \frac{2}{(-1)} = -2$$

Residue at $z = -1$ let R_2

$$R_2 = \lim_{z \rightarrow -1} (z+1) f(z) = \lim_{z \rightarrow -1} \frac{e^z + 1}{z(z-i)^2} = \frac{e^{-1} + 1}{(-1)(1+i)^2} = \frac{1 + e^{-1}}{-2i} = \frac{i}{2} \left(1 + \frac{1}{e}\right)$$

4. (a) Show that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean domain. (20)

Residue at $z=i$ R_3

$$f(z) = \frac{\phi(z)}{(z-i)^2} \quad \phi(z) = \frac{e^z + 1}{z(z+1)}$$

$$R_3 = \phi'(z=i)$$

$$= \left[\frac{e^z(z+1) - (e^z+1)(z+1)}{z^2(z+1)^2} \right]_i$$

$$= \frac{e^i(i+1)i - (e^i+1)(2i+1)}{(-1)(1+i)^2}$$

$$= \frac{e^i(-1+i-2i-1) - (2i+1)}{-2i}$$

$$= \frac{e^i(+i+2) + 2i+1}{2i}$$

$$R_3 = \frac{e^i(2i-1) + (-2+i)}{-2}$$

Now

$$\int_C f(z) dz = 2\pi i \sum_{i=1}^3 R_i$$

$$= 2\pi i \left[-2 + \frac{i}{2} \left(1 + \frac{1}{e} \right) - \frac{1}{2} [e^i(2i-1) + (i-2)] \right]$$

Candidates
must not write
on this margin

$$= 2\pi i \left\{ -2 + \frac{i}{2} + \frac{i}{2e} - ie^i + \frac{1}{2}e^i - \frac{i}{2} + 1 \right\}$$

$$= 2\pi i \left[-1 + \frac{i}{2e} - ie^i + \frac{e^i}{2} \right]$$

$$= 2\pi \left[-i - \frac{1}{2e} + e^i + \frac{ie^i}{2} \right]$$

5. (a)

Solve: $(y^2 + z^2 - x^2)p - 2xyq + 2xzr = 0$.

(10)

$$\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

$$= \frac{x dx + y dy + z dz}{-x(x + y + z)}$$

$$= \frac{\frac{1}{2} d(x^2 + y^2 + z^2)}{-x(x + y + z)}$$

Solving $\frac{dy}{-2xy} = \frac{dz}{-2xz}$

$$\log y = \log z + C_1$$

$$\Rightarrow y = z e^{C_1} \Rightarrow z = y/e^{C_1}$$

$$\frac{dx}{y^2(1 + \frac{1}{e^{2C_1}}) - x^2} = \frac{dy}{-2xy}$$

$$\frac{dx}{dy} = \frac{-1}{2x} \left(y \right) \left(1 + \frac{1}{e^{2C_1}} \right) + \frac{x}{2y}$$

let $x = vy$

$$\frac{dx}{dy} = v + y \frac{dv}{dy}$$

$$v + y \frac{dv}{dy} = -\frac{1}{2v} \left(1 + \frac{1}{e^{2C_1}} \right) + \frac{v}{2}$$

$$\frac{y dv}{dy} = -\frac{v}{2} - \frac{1}{2v} - \frac{1}{2v c_1^2}$$

$$\frac{-2v dv}{\frac{v}{2} + \frac{1}{v} \left(\frac{1}{2} + \frac{1}{c_1^2} \right)} = \frac{dy}{y}$$

$$\frac{-2v dv}{v^2 + \left(1 + \frac{1}{c_1^2}\right)} = \frac{dy}{y}$$

$$v^2 + \left(1 + \frac{1}{c_1^2}\right) = m \Rightarrow 2v dv = dm$$

$$\frac{-dm}{m} = \frac{dy}{y} \Rightarrow -\log m c_2 = \log y$$

$$y = + \frac{1}{m c_2}$$

$$y = \frac{c_2^{-1}}{v^2 + \left(1 + \frac{1}{c_1^2}\right)} = \frac{c_2^{-1}}{\left(\frac{x}{y}\right)^2 + \left(1 + \frac{1}{c_1^2}\right)}$$

$$y \left[\left(\frac{x}{y}\right)^2 + 1 + \frac{z^2}{y^2} \right] = \frac{1}{c_2}$$

$$z/y = \frac{1}{c_1}$$

$$\phi \left(\frac{z}{y}, \frac{x^2 + z^2 + y^2}{y} \right)$$

5. (b)

If $u = u(y - z, z - x, x - y)$, then find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

(10)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}$$

$$u = u(t_1, t_2, t_3)$$

$$t_1 = y - z; \quad t_2 = z - x; \quad t_3 = x - y$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1} \frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2} \frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3} \frac{\partial t_3}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_3} - \frac{\partial u}{\partial t_2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1} - \frac{\partial u}{\partial t_3}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t_2} - \frac{\partial u}{\partial t_1}$$

Adding

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

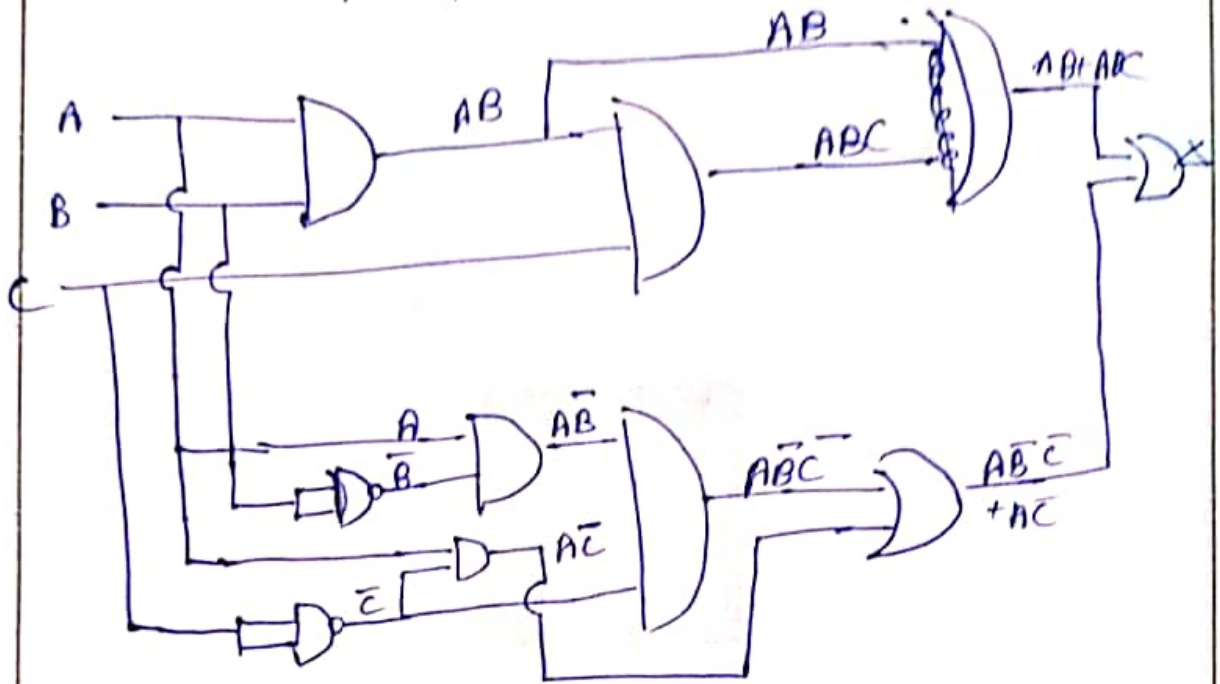
5. (c)

Given the Boolean Expression $X = AB + ABC + A\bar{B}\bar{C} + A\bar{C}$

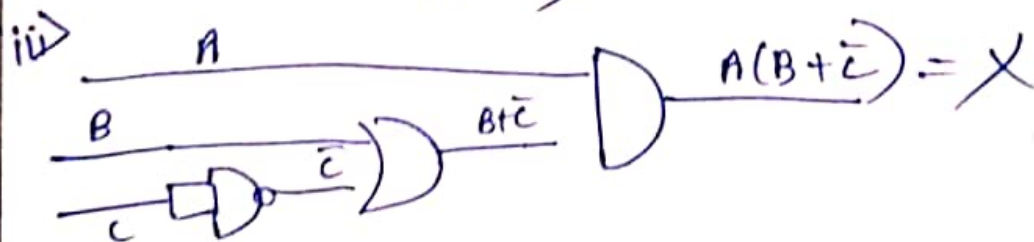
(10)

- Draw the logical diagram for the expression
- Minimize the expression
- Draw the logical diagram for the reduced expression

i) $X = AB + ABC + A\bar{B}\bar{C} + A\bar{C}$



ii)
$$\begin{aligned} X &= AB + ABC + A\bar{B}\bar{C} + A\bar{C} \\ &= AB(1 + \bar{C}) + A\bar{C}(1 + \bar{B}) \\ &= AB + A\bar{C} \\ &= A(B + \bar{C}) \end{aligned}$$



Given the velocity potential $\phi = \frac{1}{2} \log \left(\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \right)$, determine the streamlines.

$$\phi = \frac{1}{2} \left[\log((x+a)^2 + y^2) - \log((x-a)^2 + y^2) \right]$$

$$\begin{aligned} u = -\frac{\partial \phi}{\partial x} &= \frac{(x+a)}{(x+a)^2 + y^2} - \frac{(x-a)}{(x-a)^2 + y^2} \\ &= \frac{(x+a)(x-a)[x-a - x-a] + y^2(2a)}{((x+a)^2 + y^2)((x-a)^2 + y^2)} \\ &= \frac{2a[y^2 - x^2 + a^2]}{((x+a)^2 + y^2)((x-a)^2 + y^2)} \end{aligned}$$

$$\begin{aligned} v = -\frac{\partial \phi}{\partial y} &= \frac{y}{(x+a)^2 + y^2} - \frac{y}{(x-a)^2 + y^2} \\ &= \frac{y[-4ax]}{((x+a)^2 + y^2)((x-a)^2 + y^2)} \end{aligned}$$

Stream lines $\rightarrow$

$$\frac{dx}{u} = \frac{dy}{v}$$

$$\frac{dx}{2a(y^2 - x^2 + a^2)} = \frac{dy}{-4axy}$$

$$dx(-2xy) = dy(y^2 - x^2 + a^2)$$

$$(2xy) dx + (y^2 - x^2 + a^2) dy = 0$$

$$M dx + N dy = 0$$

$$\frac{\partial M}{\partial y} = 2x$$

$$\frac{\partial N}{\partial x} = -2x$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-4x}{2xy} = -\frac{2}{y}$$

$$= f(y)$$

$$IF = e^{\int f(y) dy} = e^{-2 \log y} = \frac{1}{y^2}$$

$$\left(\frac{2x}{y}\right) dx + \left(1 - \frac{x^2}{y^2} + \frac{a^2}{y^2}\right) dy = 0$$

$$M dx + N dy = 0$$

$$\frac{\partial M}{\partial y} = -\frac{2x}{y^2} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ exact

$\int M dx$ + $\int N dy$
y const term
not having x

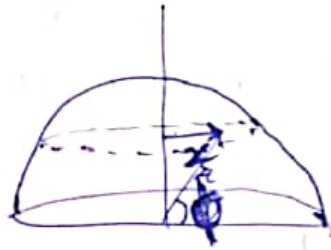
$$\frac{x^2}{y} + y - \frac{a^2}{y} = C_1$$

$$\boxed{x^2 + y^2 - a^2 = Cy}$$

streamline

5. (c)

Calculate the moment of inertia of a solid uniform hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$ with mass m about the OZ -axis. (10)



ϕ in xy plane
 θ with z axis

let ρ be mass density

$$dI = dm r^2$$

$$= \rho r dr r \sin \theta d\phi d\theta \quad \left(r \sin \theta \right)^2$$

$$I = \int_0^{\pi/2} \int_0^{2\pi} \int_0^a dI$$

$\theta=0$ $\phi=0$ $r=0$

$$= \rho \left(\int_0^{\pi/2} (\sin \theta)^3 d\theta \right) \left(\int_0^a r^2 dr \right) \left(\int_0^{2\pi} d\phi \right)$$

$$= \rho \left[\frac{2}{3} \right] \left[\frac{a^3}{3} \right] \left[2\pi \right]$$

$$= \frac{\rho (\sqrt{2}-1) \pi a^3}{10\sqrt{2}}$$

$$M = \rho \left(\frac{2}{3} \pi a^3 \right)$$

$$I = \frac{3M(\sqrt{2}-1)}{20\sqrt{2}} a^2$$

$$\sin \theta = \frac{3 \sin \theta - \sin 3\theta}{4}$$

$$\frac{\cos 3\theta - 3 \cos \theta}{4} = \frac{-\frac{1}{3\sqrt{2}} - 3\sqrt{2} + \frac{8}{3}}{4}$$

$$I = \left(\frac{9}{3} \pi a^3 \right) \left(\frac{a^2 \times 2}{5} \right)$$

$$I = M \left(\frac{2}{5} a^5 \right)$$

→ Using Newton Raphson method prove that-

¹⁹⁹⁷
(i) Iterative formula for $\frac{1}{N}$ is $x_{n+1} = x_n(2 - Nx_n)$

(ii) Iterative formula for $\frac{1}{\sqrt{N}}$ is $x_{n+1} = \frac{1}{2} \left(x_n + \frac{1}{Nx_n} \right)$

(iii) Iterative formula for $\sqrt[k]{N}$ is $x_{n+1} = \frac{1}{k} \left[(k-1)x_n + \frac{N}{x_n^{k-1}} \right]$

Solⁿ: (i) Let $x = \frac{1}{N} \Rightarrow N = \frac{1}{x}$

$$\Rightarrow \frac{1}{x} - N = 0$$

$$\text{Let } f(x) = \frac{1}{x} - N$$

Different $f(x)$ is there for these iterations, how to guess $f(x)$

$$\text{eg } f(x) = \frac{1}{x} - N, \quad x = \frac{1}{N}$$

for $x = 1/N$

7. (a)

Find the equations of the system of curves on the cylinder $2y = x^2$ orthogonal to its intersections with the hyperboloids of one-parameter system $xy = z + c$. (15)

To find intersection of given curve and hyperboloid.

$$xy = z + C \left(\frac{2y}{x^2} \right)$$

$$x^3y = x^2z + 2yC \quad \text{--- i}$$

Differentiating

$$3x^2y dx + x^3 dy = 2xz dx + x^2 dz + 2dyC$$

putting C in i

$$x^3y = x^2z + \frac{2y \left[3x^2y dx + x^3 dy - 2xz dx - x^2 dz \right]}{2dy}$$

$$x^2z dy + 3x^2y^2 dx - \cancel{x^3y dx} - 2xyz dx - x^2y dz = 0$$

$$xz dy + 3xy^2 dx - 2yz dx - xy dz = 0$$

$$dy - x dx = 0$$

$$-x dx + dy + 0 dz = 0$$

$$(3xy^2 - 2yz) dx + xz dy - xy dz = 0$$

Candidates
must not write
on this margin

$$\frac{dx}{-xy} = \frac{-dy}{x^2y} = \frac{dz}{-x^2z - 3xy^2 + 2yz}$$

$$x dx = dy$$

$$\frac{x^2}{2} = y + C_1$$

$$\frac{dx}{-x\left(\frac{x^2}{2}\right)} = \frac{dz}{-\cancel{x^2z} - 3x\left(\frac{x^4}{4}\right) + \cancel{\frac{2x^2}{2}z}}$$

$$dx \left(\frac{-3x^5}{4} \right) = dz \left(\frac{-x^3}{2} \right)$$

$$dx \left(\frac{3}{2} x^2 \right) = dz$$

$$\frac{3}{2} x^3 + C_2 = z$$

$$\phi \left(\frac{3}{2} x^3 - z, \frac{x^2}{2} - y \right) = 0$$

For an integral $\int_{-1}^1 f(x) dx$, show that two-point Gauss Quadrature Rule is given

by $\int_{-1}^1 f(x) dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$. Using this rule, estimate $\int_2^4 2xe^x dx$.

(20)

$$\int_{-1}^1 f(x) dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

$$= w_1 f(x_1) + w_2 f(x_2)$$

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$\int_{-1}^1 f(x) dx = 2a_0 + \frac{2}{3} a_2$$

$$= w_1 f(x_1) + w_2 f(x_2)$$

$$= a_0(w_1 + w_2) + a_1(w_1 x_1 + w_2 x_2) + a_2(w_1 x_1^2 + w_2 x_2^2)$$

Comparing

$$w_1 + w_2 = 2$$

$$w_1 x_1 + w_2 x_2 = 0$$

$$w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3}$$

$$w_1 x_1^3 + w_2 x_2^3 = 0$$

$$w_1 = w_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}$$

$$\therefore \int_{-1}^1 f dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$\int_2^4 2x e^x dx$$

$$\begin{array}{r} -1 \rightarrow 2 \\ 1 \rightarrow 4 \end{array}$$

$$(y-f) = \frac{y-1}{2-(-1)} (x-1)$$

$$u = ax + b$$

$$-1 = 2a + b$$

$$1 = 4a + b$$

$$-2 = -2a \Rightarrow a = 1 ; b = -3$$

$$u = x - 3$$

$$x = u + 3$$

$$I = \int_{-1}^1 2(u+3) e^{u+3} du$$

$$I = 2e^3 \int_{-1}^1 (u+3) e^u du$$

$$\int_{-1}^1 f(u) du = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$I = 2e^3 \left[\left(3 - \frac{1}{\sqrt{3}}\right) e^{-\frac{1}{\sqrt{3}}} + \left(3 + \frac{1}{\sqrt{3}}\right) e^{\frac{1}{\sqrt{3}}} \right]$$

(c)

Candidates
must not write
on this margin

Consider a system with two degrees of freedom for which $V = q_1^2 + 3q_1q_2 + 4q_2^2$.
Find the equilibrium position and determine if the equilibrium position is stable.
(15)

$$\frac{\partial V}{\partial q_1} = 2q_1 + 3q_2 = 0$$

$$\frac{\partial V}{\partial q_2} = 3q_1 + 8q_2 = 0$$

$$\Rightarrow q_1 = q_2 = 0$$

$$\frac{\partial^2 V}{\partial q_1^2} = 2 ; \quad \frac{\partial^2 V}{\partial q_2^2} = 8$$

$$\frac{\partial^2 V}{\partial q_1 \partial q_2} = 3$$

$$\Delta = 16 - 9 > 0$$

$\Rightarrow$ minimum

$\Rightarrow$ stable equilibrium.