

NEXT IAS

(To be filled by candidate)

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Exam Centre : Old Rajinder Nagar ☐ Online ☒

CSE (MAINS) TEST SERIES-2024

Test ~~009~~ 11	MATHEMATICS
Dated : 14-07-2024	OPTIONAL

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are EIGHT questions divided in Two sections and printed in ENGLISH.

Candidate has to attempt FIVE questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question cum Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1 (a)	4	10			
1 (b)	6	10			
1 (c)	8	10			
1 (d)	10	10			
1 (e)	12	10			
2 (a)	14	15			
2 (b)	17	15			
2 (c)	20	20			
3 (a)	24	20			
3 (b)	28	15			
3 (c)	31	15			
4 (a)	34	15			
4 (b)	37	15			
4 (c)	40	20			
5 (a)	44	10			
5 (b)	46	10			
5 (c)	48	10			
5 (d)	50	10			
5 (e)	52	10			
6 (a)	54	20			
6 (b)	58	15			
6 (c)	61	15			
7 (a)	64	15			
7 (b)	67	20			
7 (c)	71	15			
8 (a)	74	15			
8 (b)	77	15			
8 (c)	80	20			
Grand Total					

Remarks

Observations:

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SECTION-A

1. (a) Show that any two eigen vectors corresponding to the distinct eigenvalues of a unitary matrix are orthogonal. (10)

let λ_1, λ_2 be distinct eigen values.

x_1, x_2 be eigen vectors

$$AA^{\theta} = I$$

$$Ax_1 = \lambda_1 x_1$$

$$Ax_2 = \lambda_2 x_2$$

$$(Ax_2)^{\theta} = (\lambda_2 x_2)^{\theta}$$

$$(x_2^T A^T)^{\theta} = (x_2^T \lambda_2)^{\theta}$$

$$x_2^{\theta} A^{\theta} = x_2^{\theta} \overline{\lambda_2}$$

$$x_2^{\theta} A = x_2^{\theta} \overline{\lambda_2}$$

$$(x_2^{\theta} A) x_1 = (x_2^{\theta} \overline{\lambda_2}) x_1$$

$$x_2^{\theta} (\lambda_1 x_1) = (x_2^{\theta} x_1) \overline{\lambda_2}$$

$$(x_2^{\theta} x_1) (\lambda_1 - \overline{\lambda_2}) = 0$$

$\therefore \lambda_1 \neq \overline{\lambda_2}$ and λ_1, λ_2 distinct

$\therefore x_2^{\theta} x_1 = 0$ i.e. orthogonal

1. (b)

Find the sum and product of the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{bmatrix} \quad (10)$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 & 4 \\ 2 & 1-\lambda & 5 & 6 \\ 7 & 4 & 3-\lambda & 2 \\ 4 & 3 & 0 & 5-\lambda \end{vmatrix} = 0$$

expanding along 4th row.

$$-4 \left(\begin{vmatrix} 2 & 3 & 4 \\ 1-\lambda & 5 & 6 \\ 4 & 3-\lambda & 2 \end{vmatrix} \right) + 3 \begin{vmatrix} 1-\lambda & 3 & 4 \\ 2 & 5 & 6 \\ 7 & 3-\lambda & 2 \end{vmatrix}$$

$$+ (5-\lambda) \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 1-\lambda & 5 \\ 7 & 4 & 3-\lambda \end{vmatrix}$$

$$= -4 \left[2(10 - 18 + 6\lambda) - 3(2 - 2\lambda - 24) + 4(\lambda - 3)\lambda - 1 \right] - 20$$

$$+ 3 \left[(1-\lambda)(10 - 18 + 6\lambda) - 3(-38) + 4(-29 - 2\lambda) \right]$$

$$+ (5-\lambda) \left[(1-\lambda)(3 - 4\lambda + \lambda^2 - 20) - 2(6 - 2\lambda) \right] - 35$$

$$+ 3(8 - 7 + 7\lambda)$$

$$= -4 \left[\cancel{-18} + 2\lambda + 4\lambda^2 \right] \\ + 3 \left[-10 + 6\lambda - 6\lambda^2 \right] \\ + (5-\lambda) \left[44 + 38\lambda + 5\lambda^2 - \lambda^3 \right]$$

$$\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta = 0$$

$$\delta = 262 = \text{product of eigen values}$$

$$\gamma = 156 = \text{sum of eigen values}$$

1. (c)

If $u = z \sin\left(\frac{y}{x}\right)$; where $x = 3r^2 + 3s$, $y = 4r - 2s^3$, $z = 2r^2 - 3s^2$, then find $\frac{\partial u}{\partial r}$ and

$$\frac{\partial u}{\partial s}$$

(10)

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r}$$

$$= \left(3 \cos \frac{y}{x}\right) \left(\frac{y}{x^2}\right) [6r] + \left[\frac{3 \cos \frac{y}{x}}{x}\right] [4]$$

$$+ \left[\sin \frac{y}{x}\right] [4r]$$

$$= \cos \frac{4r-2s^3}{3r^2+3s} \left(\frac{4(2r^2-3s^2)}{3r^2+3s} \right) + \frac{6r(2r^2-3s^2)}{3r^2+3s} + \sin \frac{4r-2s^3}{3r^2+3s} [4r]$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}$$

$$= \left(\frac{-3y}{x^2} \cos \frac{y}{x}\right) 3 + \left(\frac{3 \cos \frac{y}{x}}{x}\right) (-6s^2) + \left(\sin \frac{y}{x}\right) (-6s)$$

1. (d)

Evaluate, $\lim_{x \rightarrow \infty} \left\{ \frac{\log x}{x} \right\}^{1/x}$

(10)

$$\text{let } L = \lim_{x \rightarrow \infty} \left\{ \frac{\log x}{x} \right\}^{1/x}$$

$$\log L = \lim_{x \rightarrow \infty} \frac{1}{x} \log \frac{\log x}{x} \quad \left(\frac{-\infty}{\infty} \right)$$

$$= \frac{x}{\log x} \left(\frac{1}{x^2} - \frac{\log x}{x^2} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{1 - \log x}{x \log x} \quad \left(\frac{\infty}{\infty} \right)$$

$$= \lim_{x \rightarrow \infty} -\frac{1}{x(\log x + 1)}$$

$$= 0$$

$$\log L = 0$$

$$L = 1$$

1. (c)

Find the equation of the sphere passing through the points $(1,1,2)$, $(1,-1,2)$ and $(1,1,3)$ having centre on the line $x+y-z-1=0=2x+y-z-2$. (10)

Let α, β, γ be centre.

$$\alpha + \beta - \gamma - 1 = 0 \quad \text{e}$$

$$2\alpha + \beta - \gamma - 2 = 0$$

$$(\alpha-1)^2 + (\beta-1)^2 + (\gamma-2)^2 = (\alpha-1)^2 + (\beta+1)^2 + (\gamma-3)^2$$

$$\Rightarrow \beta = 0$$

$$\alpha - \gamma = 1$$

$$2\alpha - \gamma = 2$$

$$\alpha = 1 ; \gamma = 0$$

$$\text{radius}^2 = 5$$

$$(x-1)^2 + (y-0)^2 + (z-0)^2 = 5^2$$

3. (a)

Diagonalise the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ through an orthogonal transformation.

(20)

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) [\lambda^2 - 5\lambda + 6] + (\lambda - 3) = 0$$

$$(\lambda - 3) [(\lambda - 2)(2 - \lambda) + 1] = 0$$

$$(\lambda - 3) [-\lambda^2 - 3 + 4\lambda] = 0$$

$$(\lambda - 3) (\lambda^2 - 4\lambda + 3) = 0$$

$$(\lambda - 3) (\lambda - 3) (\lambda - 1) = 0$$

$$\lambda = 3, 3, 1$$

$$AX = \lambda X$$

$$\boxed{\lambda = 3}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3 = x_1$$

$$\text{let } x_3 = x_1 = k_1$$

$$x_2 = k_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 2 & 0 & x_2 \\ 1 & 0 & 1 & x_3 \end{array} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0 \quad ; \quad x_2 = 0$$

$$x_1 = -x_3 = k \quad ; \quad x_2 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ -k \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ are}$$

orthogonal

therefore orthonormal set is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}; \quad P P^T = I$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P^{-1} = P^{-1}$$

$$P^{-1} A P =$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} & 0 & \frac{3}{\sqrt{2}} \\ 0 & 3 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} P$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. (b)

Given $xy = c$, where c is constant. Show that the minimum value of $(x+y)$ occurs, when $x = y$. Deduce that the minimum value of $a \cot \theta + b \tan \theta$ is $2\sqrt{ab}$ where a and b are constants. (15)

$$Z: x+y$$

$$\text{sc } xy = c$$

$$\Rightarrow y = c/x$$

$$Z: x + c/x$$

$$\frac{dz}{dx} = 1 - \frac{c}{x^2} = 0$$

$$\Rightarrow x = \pm \sqrt{c}$$

$$\Rightarrow y = \pm \sqrt{c}$$

$$x+y = \pm 2\sqrt{c}$$

$$\frac{d^2z}{dx^2} = \frac{2c}{x^3} > 0 \text{ at } x = \sqrt{c}$$

hence $x = y = \sqrt{c}$ is ~~min~~ ^{min}

$$\Rightarrow \min(x+y) = 2\sqrt{c}$$

when $x = y$

for $a \cot \theta + b \tan \theta$

$$x = a \cot \theta$$

$$y = b \tan \theta$$

$$xy = ab$$

$$\Rightarrow \text{min at } x = y = \sqrt{ab}$$

$$a \cot \theta = b \tan \theta = \sqrt{ab}$$

$$\begin{aligned} \text{min } (a \cot \theta + b \tan \theta) &= \sqrt{ab} + \sqrt{ab} \\ &= 2\sqrt{ab} \end{aligned}$$

Prove that the centres of the spheres which touch the lines $y = mx$, $z = -c$, $y = -mx$, $z = c$ lie upon the conicoid $mxy + cz(1 + m^2) = 0$. (15)

Let centre is (α, β, γ)

Let radius is r

$$L_1 \equiv \frac{x}{1} = \frac{y}{m} = \frac{z-c}{0}$$

length of perpendicular from (α, β, γ) to L_1 is r .

$$\Rightarrow r^2(1 + m^2 + 0^2) = \left\{ (m\alpha - \beta)^2 + (m)^2(\gamma - c)^2 + (\gamma - c)^2 \right\} \quad \text{--- i}$$

Also

$$L_2 \equiv \frac{x}{1} = \frac{y}{-m} = \frac{z+c}{0}$$

$$r^2(1 + m^2 + 0^2) = \left\{ (-m\alpha - \beta)^2 + (-m)^2(\gamma + c)^2 + (\gamma + c)^2 \right\} \quad \text{--- ii}$$

Dividing i and ii

$$(m\alpha - \beta)^2 + m^2(\gamma - c)^2 + (\gamma - c)^2 = (m\alpha + \beta)^2 + m^2(\gamma + c)^2 + (\gamma + c)^2$$

$$4m\alpha\beta + (m^2 + 1)(\gamma - c)(\gamma + c) = 0$$

$$(m^2 + 1)cz + mxy = 0$$

8. (a)

Find the Laplace transform of $\tan^{-1} \frac{1}{s}$

$$\tan^{-1} \frac{1}{s} = \frac{1}{s} - \frac{1}{3} \left(\frac{1}{s} \right)^3 + \frac{1}{5} \left(\frac{1}{s} \right)^5 - \frac{1}{7} \left(\frac{1}{s} \right)^7 + \dots$$

$$L(\tan^{-1} \frac{1}{s}) = L\left(\frac{1}{s}\right) - \frac{1}{3} L\left(\frac{1}{s^3}\right) + \frac{1}{5} L\left(\frac{1}{s^5}\right) - \dots$$

$$L(f(t)) = \dots \quad L(t f(t)) = -f'(s)$$

$$L(\tan^{-1} \frac{1}{s}) = \dots \quad L(f'(t)) = sf(s) - f(0)$$

$$L\left(\frac{1}{1 + \frac{1}{t^2}}\right) = s L\left(\tan^{-1} \frac{1}{s}\right) - \frac{\pi}{2}$$

$$L\left(\frac{t^2}{t^2 + 1}\right) = s L\left(\tan^{-1} \frac{1}{s}\right) - \frac{\pi}{2}$$

$$L\left(1 - \frac{1}{t^2 + 1}\right) = s L\left(\tan^{-1} \frac{1}{s}\right) - \frac{\pi}{2}$$

$$\frac{1}{s} -$$

5. (b)

Determine the constant 'a' so that the vector $V = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal.

(10)

$$\nabla \cdot V = 0$$

$$1 + 1 + a = 0$$

$$a = -2$$



5. (c)

Evaluate $\iint_S \mathbf{r} \cdot \mathbf{n} dS$, where S is a closed surface.

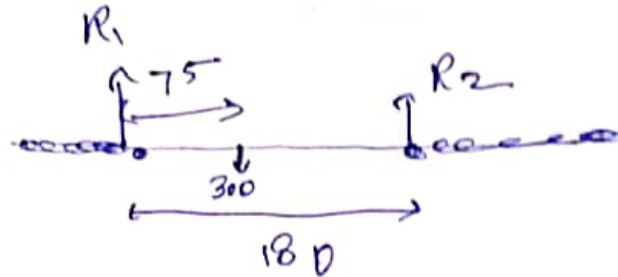
(10)

$$\begin{aligned}\iint_S \mathbf{r} \cdot \mathbf{n} dS &= \iiint_V \nabla \cdot \mathbf{r} dV \\ &= 3 \iiint_V dV \\ &= 3V\end{aligned}$$

where V is volume enclosed by closed surface S .

5. (d)

A horizontal rod, 1.8 m long, whose weight may be neglected rests on two supports at its extremities, a body of weight 300kg is suspended from the rod at a distance of 75 m from one end; find the reaction at each point of support if one support could only bear a thrust equal to the weight of 50kg, what is the greatest distance from the other support at which the body could be suspended? (10)



$$R_1 + R_2 = 300$$

$$R_1(75) = R_2(180 - 75)$$

$$= 105 R_2$$

$$R_2 \left(1 + \frac{105}{75}\right) = 300$$

$$R_2 = \frac{300 \times 75}{180} = 125$$

$$R_1 = 175$$

Let R_1 support only 50 kg thrust

$$x(50) = R_2(180 - x)$$

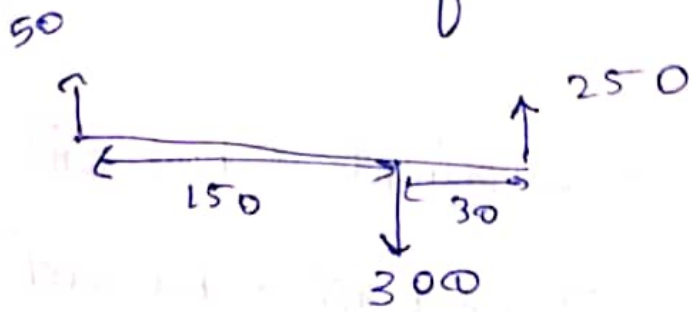
$$R_1 + R_2 = 300$$

$$R_2 = 250$$

$$x = \frac{5(180 - x)}{1}$$

$$\boxed{x = 150}$$

distance from other support = 30



5. (c)

Show that if the displacements of a particle in a straight line is expressed by the equation $x = a \cos nt + b \sin nt$, it describes a simple harmonic motion whose amplitude is $\sqrt{a^2 + b^2}$ and period $2\pi/n$ (10)

$$\dot{x} = -an \sin nt + bn \cos nt$$

$$\ddot{x} = -n^2 [a \cos nt + b \sin nt]$$

$$\ddot{x} = -n^2 x$$

therefore SHM.

$$\text{period} = \frac{2\pi}{n} \left(\frac{2\pi}{\omega} \right) \left(\ddot{x} = -\omega^2 x \right)$$

$$\text{do } \dot{x} = 0$$

$$\Rightarrow b \cos nt = a \sin nt$$

$$\tan nt = b/a$$

$$nt = \frac{1}{n} \tan^{-1} \left(\frac{b}{a} \right)$$

$$x = a \cos \tan^{-1} \frac{b}{a} + b \sin \tan^{-1} \frac{b}{a}$$

$$= \frac{a}{\sec \tan^{-1} \frac{b}{a}} + b \sqrt{1 - \cos^2 \tan^{-1} \frac{b}{a}}$$

$$= \frac{a}{\sqrt{1 + \left(\frac{b}{a}\right)^2}} + b \sqrt{1 - \left(\frac{1}{\sqrt{1 + \left(\frac{b}{a}\right)^2}}\right)^2}$$

$$= \frac{a^2}{\sqrt{a^2 + b^2}} + b \sqrt{\frac{b^2/a^2}{a^2 + b^2}}$$

Candidates must not write on this margin

$$= \frac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2}$$

6. (a)

Obtain the Clairaut's form of the differential equation

$$\left(x \frac{dy}{dx} - y\right) \left(y \frac{dy}{dx} + x\right) = a^2 \frac{dy}{dx}. \text{ Also find its general solution.}$$

[20]

$$(xp - y)(yp + x) = a^2 p$$

$$(xp - y) = \frac{a^2 p}{yp + x}$$

$$y = xp - \frac{a^2 p}{yp + x}$$

$$x^2 + y^2 = u$$

$$2(x + yp) = \frac{du}{dx}$$

$$p = \left(\frac{1}{2} \frac{du}{dx} - x\right) \frac{1}{y}$$

$$y = x \left[\frac{1}{2} \frac{du}{dx} - x \right] \frac{1}{y} - a^2 \left(\frac{1}{2} \frac{du}{dx} - x \right) \frac{1}{y}$$

$$y^2 = \frac{x}{2} \frac{du}{dx} - x^2 - \frac{a^2}{y} + \frac{a^2 x}{2 \frac{du}{dx}}$$

$$x^2 + y^2 = u = \frac{x}{2} p - \frac{a^2}{y} + \frac{a^2 x}{2p}$$

$$u = \frac{1}{2} \left[xp + \frac{a^2 x}{2p} - \frac{a^2}{y} \right]$$

$$x^2 + y^2 = u \quad ; \quad \frac{y}{x} = v$$

$$2\left(x + y \frac{dy}{dx}\right) = \frac{du}{dx}$$

$$\frac{x \frac{dy}{dx} - y}{x^2} = \frac{dv}{dx}$$

$$\frac{du}{dv} = \frac{2(x dx + y dy)}{x dy - y dx} \cdot x^2 = \frac{2x^2 \left[1 + \frac{y}{x} \frac{dy}{dx}\right]}{\left[\frac{dy}{dx} - \frac{y}{x}\right]}$$

$$x^2 + v^2 x^2 = u \Rightarrow x^2 = \frac{u}{1 + v^2}$$

$$\Rightarrow y^2 = \frac{uv^2}{1 + v^2}$$

$$\left(x^2 \frac{dv}{dx}\right) \left(\frac{1}{2} \frac{du}{dx}\right) = a^2 \left[\frac{dy}{dx}\right]$$

$$\left[\frac{du}{dv} (2x^2)\right] \left[p - \frac{y}{x}\right] = 1 + \frac{y}{x} p$$

$$p \left[\frac{p}{2x^2} - \frac{y}{x}\right] = 1 + \frac{y}{x} p$$

$$p = \frac{1 + \frac{v(1+v^2)}{2u} p}{\frac{p(1+v^2)}{2u} - v}$$

$$\frac{x^2}{2} \frac{dv}{dx} \frac{du}{dx}$$

$$xy p^2 + x^2 p - y^2 p - xy = a^2 p$$

$$y^2 p = xy p^2 + x^2 p - xy - a^2 p$$

$$y^2 = xy p + x^2 - \frac{xy}{p} - a^2$$

$$y^2 = p \left(\frac{y}{x} \right)^2 x^2$$

$$(xp - y)(yp + x) = a^2 p$$

$$(xp - y) \left(\frac{py}{x} + 1 \right) = a^2 \frac{p}{x}$$

$$(xpy - y^2) \left(1 + \frac{py}{x} \right) = a^2 \frac{py}{x}$$

$$xpy - y^2 = \frac{a^2 py/x}{1 + py/x}$$

$$y^2 = xpy - \frac{a^2 py/x}{1 + py/x}$$

$$x^2 = x; \quad y^2 = y$$

$$p = \frac{dy}{dx} = \frac{y}{x} \frac{dy}{dx} = \frac{y}{x} p$$

$$y = x^2 p - \frac{a^2 p}{1 + p}$$

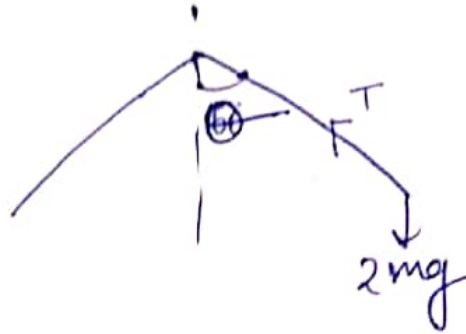
$$y = xp - \frac{a^2 p}{1 + p}$$

$$y = xc - \frac{a^2 c}{1 + c}$$

$$y^2 = x^2 c - \frac{a^2 c}{1 + c}$$

6. (b)

Show that the greatest angle through which a person can oscillate on a swing, the ropes of which can support twice the person's weight at rest is 120° . If the ropes are strong enough and he can swing through 180° and if v is his speed at any point, prove that the tension in the rope at that point is $\frac{3mv^2}{2l}$ where m is the mass of the person and l the length of the rope. (15)



$$T \cos \theta = 2mg \Rightarrow T = 2mg \quad \left(\because \frac{mv^2}{l} = 0 \text{ as } v = 0 \right)$$

$$T = 2mg$$

$$\cos \theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$$



$$T - mg \cos \theta = \frac{mv^2}{l}$$

also

$$m \frac{d^2 s}{dt^2} = -mg \sin \theta$$

$$s = l\theta$$

$$ml \frac{d^2 \theta}{dt^2} = -mg \sin \theta$$

$$\frac{d^2\theta}{dt^2} = -g \sin \theta$$

$$\frac{d\theta}{dt} = \sqrt{g} \cos \theta$$

$$(\dot{\theta})^2 = +2g \cos \theta + A$$

at $\theta = \pi/2$; $\dot{\theta} = 0 \Rightarrow A = 0$

$$\dot{\theta} = \sqrt{2g \cos \theta}$$

$$\frac{d\theta}{dt} = \sqrt{2g} \sqrt{\cos \theta}$$

$$\frac{d\theta}{\sqrt{\cos \theta}} = \sqrt{2g} dt$$

$$T = mg \cos \theta + \frac{m l^2 \dot{\theta}^2}{l}$$

$$= m(g \cos \theta + l \cdot 2g \cos \theta)$$

$$T = 3mg \cos \theta$$

$$v = l \dot{\theta} = \sqrt{2gl \cos \theta}$$

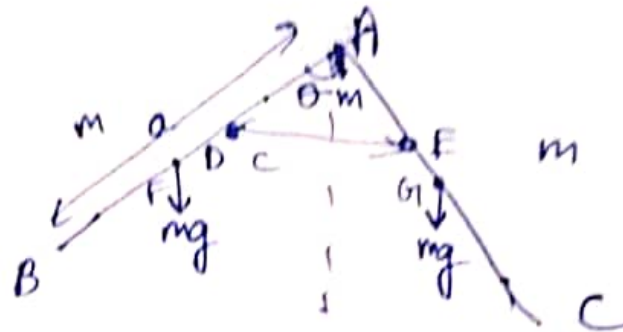
$$\Rightarrow \cos \theta = \frac{v^2}{2gl}$$

$$\Rightarrow T = 3mg \frac{v^2}{2gl}$$

$$T = \frac{3m}{2l} v^2$$

Two equal uniform rods AB and AC , of length a each, are freely joined at A , and are placed symmetrically over two smooth pegs on the same horizontal level at a distance c apart ($3c < 2a$). A weight equal to that of a rod is suspended from the joint A . In the position of equilibrium, find the inclination of either rod with the horizontal by the principle of virtual work.

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$$\triangle ABC \sim \triangle ADE$$

$$\frac{AD}{AB} = \frac{DE}{BC}$$

$$AD = \frac{a(c)}{2a \sin \theta} = \frac{c}{2 \sin \theta}$$

$$c < \frac{2a}{3}$$

$$AD < \frac{a}{3 \sin \theta}$$

$$AF = a/2 \Rightarrow DF = \frac{a}{2} - \frac{c}{2 \sin \theta}$$

$$DF \cos \theta = \left(\frac{a}{2} - \frac{c}{2 \sin \theta} \right) \cos \theta$$

$$-mg \delta \left(\frac{c \cos \theta}{2 \sin \theta} \right) + 2mg \delta \left(\frac{a \cos \theta}{2} - \frac{c \cos \theta}{2 \sin \theta} \right) = 0$$

$$- \frac{c}{2} (-\cos^2 \theta) + 2 \left[-\frac{a \sin \theta}{2} + \frac{c \cos^2 \theta}{2} \right] = 0$$

$$\frac{3c}{2} = a \sin^3 \theta \Rightarrow \sin^3 \theta = \frac{3c}{2a}$$

$$\theta = \sin^{-1} \left(\left(\frac{3c}{2a} \right)^{1/3} \right)$$

7. (a)

Find the solution of the differential equation $2x^2y \left(\frac{dy}{dx} \right) - \tan^2(x^2y^2) = 2xy^3$. What will be the definite value of the arbitrary constant, appearing in the solution, on coordinate axes?

$$x^2y^2 = u$$

$$2xy^2 + 2x^2y \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{du}{dx} = \tan^2 u$$

$$\frac{du}{\tan^2 u} = dx$$

$$\frac{\cos^2 u}{\sin^2 u} du = dx$$

$$(\sec^2 u - 1) du = dx$$

$$-\cot u - u = x + C_1$$

$$-\cot x^2y^2 - x^2y^2 = x + C_1$$

On coordinate axes.

$$-1 - x^2y^2 \tan x^2y^2 = x \tan x^2y^2 + C_1 \tan x^2y^2$$

~~On axes~~

$$C_1 = \frac{-1}{\tan x^2y^2}$$

Verify Stokes theorem for $\vec{F} = x\hat{i} + z^2\hat{j} + y^2\hat{k}$ over the plane surface: $x+y+z=1$, $x+y+z=1$, lying in the first octant. (20)

$$\oint \vec{F} \cdot \hat{n} \, ds =$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds = \oint_C \vec{F} \cdot d\vec{r}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & z^2 & y^2 \end{vmatrix}$$

$$= \hat{i}(2y - 2z) - \hat{j}$$

$$\hat{n} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$$

$$\nabla \times \vec{F} \cdot \hat{n} = \frac{2y - 2z}{\sqrt{3}}$$

$$ds = \frac{dx \, dy}{\hat{n} \cdot \hat{k}} = \sqrt{3} \, dx \, dy$$

$$\iint_R (2y - 2z) \, dx \, dy$$

$$\iint_{y=0, x=0}^{1-y} (4y - 2 + 2x) \, dx \, dy$$

$$\int_0^1 (4y(1-y) - 2(1-y) + (1-y)^2) \, dy$$

$$= \int_0^1 (4y - 4y^2 - 2 + 2y + 1 - 2y + y^2) \, dy$$

$$= \int_0^1 (4y - 3y^2 - 1) \, dy = 2 - 1 - 1 = 0$$

$$\oint F \cdot dr$$

$$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$x + y + z = 1$$

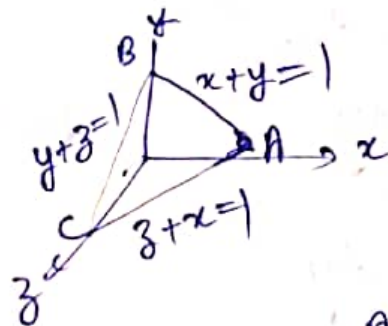
$$dx + dy + dz = 0$$

$$\oint x dx + z^2 dy + y^2 dz$$

$$\oint x dx + (1-x-y) dy + y^2 (dx - dy)$$

$$\oint x dx + dy + z^2 dy - 2x dy - 2y dy + 2xy dy - y^2 dx$$

$$\oint x^2 dy - 2x dy + 2xy dy - y^2 dx$$



$$AB \rightarrow dy = -dx$$

$$y = 1 - x$$

$$\int_{AB} x^2 (-dx) - 2x(-dx) + 2(1-y)y dy + y^2 dy$$

$$+ \int_{BC} (x=0, dx=0) + \int_{CA} (y=0, dy=0)$$

$$-\frac{1}{3} + 1 + \frac{2}{3} - \frac{1}{3} = \frac{4}{3}$$

$$AB \rightarrow dz = z = 0$$

$$\int_{AB} x dx = -1/2$$

$$BC \rightarrow dx = x = 0; dy = -dz$$

$$\int_{BC} -z^2 dz - y^2 dy = -2/3$$

$$CA \rightarrow -\int_{z=0} z^2 dz - \int_{y=1} y^2 dy = 0$$

$$CA \rightarrow y = dy = 0$$

$$\int_0^1 x dx = 1/2$$

$$AB + BC + CA = \boxed{0}$$

$$\underline{LHS = RHS}$$

If $F = \mu(u^2 - au^3)$, where $a > 0$, be the central force per unit mass and a particle is projected from an apse at a distance a from the centre of force with a velocity

$\sqrt{\frac{\mu c}{a^2}}$, where $a > c$, prove that the other apsidal distance of the orbit is $\frac{a(a+c)}{a-c}$.

(15)

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2 u^2}$$

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu(1-au)}{h^2}$$

Multiplying by $2 \frac{du}{d\theta}$ and integrating with θ

$$\left(\frac{du}{d\theta}\right)^2 + \frac{u^2}{2} = \frac{2\mu u}{h^2} - \frac{2\mu a u^2}{2h^2} + \frac{C_1}{h^2}$$

at apse $\frac{du}{d\theta} = 0$

$$r = a = \frac{1}{u}$$

$$v = \sqrt{\mu c u^2} = u \sqrt{\mu c}$$

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2 \right] = \mu u - \frac{\mu a u^2}{2} + C_1$$

$$u^2 \mu c = h^2 u^2 = 2\mu u - \frac{2\mu a u^2}{2} + C_1$$

$$\frac{\mu c}{a^2} = \frac{h^2}{a^2} = 2\frac{\mu}{a} - \frac{2\mu}{2a^2} + C_1$$

$$\Rightarrow h = \sqrt{\mu c}$$

$$\Rightarrow \mu c \left[u^2 + \left(\frac{du}{d\theta}\right)^2 \right] = 2\mu u - \frac{2\mu a u^2}{2} + \frac{\mu c}{a^2} - \frac{2\mu}{a} + \frac{2\mu}{2a^2}$$

At another apse $\frac{du}{d\theta} = 0$

$$u^2 = 2uu - \frac{u^2(2a)}{2} + \frac{uc}{a^2} - \frac{2u}{a} + \frac{2u}{2a^2}$$

$$u^2(c + \frac{2a}{2}) - 2u - \frac{c}{a^2} + \frac{2 \cdot 1}{a} - \frac{2 \cdot 1}{2a^2} = 0$$

Quadratic $\Rightarrow u = \alpha, \beta$

$$\alpha + \beta = \frac{2}{c + \frac{2a}{2}}$$

$$\alpha = \frac{1}{a}$$

$$\Rightarrow \beta = \frac{2}{2c + a} - \frac{1}{a} = \frac{a - c}{a(2c + a)}$$

~~$$\frac{1}{a} \beta = \frac{\frac{1}{a} - \frac{c}{2a^2} - \frac{1}{a^2}}{c + \frac{1}{2}}$$~~

~~$$\beta = \frac{1 - \frac{1}{2a} - \frac{1}{a}}{c + \frac{1}{2}}$$~~

Other apsidal distance is

~~$$\frac{2a - 2c - 1}{a(2c + 1)}$$~~

~~$$\frac{1}{a} \beta = \frac{-\frac{c}{a^2} + \frac{1}{2a}}{c + \frac{1}{2}}$$~~

~~$$\beta = \frac{a - 2c}{(2c + 1)a^2}$$~~

$$2a - 2c - 1 = a - 2c$$

$$\alpha\beta = \frac{-\frac{c}{a^2} + \frac{1}{a}}{c+a}$$

$$\beta = \frac{-\frac{c}{a} + 1}{c+a} = \frac{a-c}{a(a+c)}$$

$$r_2 = \frac{a(a+c)}{a-c}$$