

(To be filled by candidate)

Name : Madhukant

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Exam Centre : Old Rajinder Nagar ☐ Bhopal ☐ Jaipur ☐Mukherjee Nagar ☐ Prayagraj ☐ Online ☒

CSE (MAINS) TEST SERIES-2024

Test-10

MATHEMATICS OPTIONAL

Dated : 18-08-2024

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

*(Please read each of the following instructions carefully before attempting questions)**There are **EIGHT** questions divided in **Two sections** and printed in **ENGLISH**.**Candidate has to attempt **FIVE** questions in all.****Question No. 1 and 5** are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.**The number of marks carried by a question/part is indicated against it.**Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.**No marks will be given for answers written in a medium other than the authorized one.**Word limit in questions, wherever specified, should be adhered to.**Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.***DELHI CENTRE:****Old Rajinder Nagar:** 27-B, Pusa Road, Metro Pillar no. 118, Near Karol Bagh Metro, New Delhi-110060 | Ph: 8081300200**Mukherjee Nagar:** 1422, Main Mukherjee Nagar Road, Near Batra Cinema, New Delhi-110009 | Ph: 8081300200**BHOPAL CENTRE:** Plot No. 46 Zone - 2, M.P Nagar, Bhopal - 462011 | Ph: 8827664612, 8081300200**JAIPUR CENTRE:** 6 & 7, 3rd Floor, Sree Gopal Nagar, Gopalpura Bypass, Jaipur - 302015 | Phone: 9358200511**PRAYAGRAJ CENTRE:** 31/31 Sardar Patel Marg, Civil Lines, Prayagraj, Uttar Pradesh - 211001 | Phone: 9958857757

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Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	15			
2. (c)	20	20			
3. (a)	24	20			
3. (b)	28	15			
3. (c)	31	15			
4. (a)	34	20			
4. (b)	38	15			
4. (c)	41	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	15			
7. (c)	70	20			
8. (a)	74	15			
8. (b)	77	20			
8. (c)	81	15			
Grand Total					

Remarks

Observations:

SECTION-A

1. (a) Show that $(1+i)$ is prime element in the ring R of Gaussian integers. (10)
- (b) If H is a subgroup of a group G such that $x^2 \in H$ for every $x \in G$, then prove that H is a normal subgroup of G . (10)
- (c) Find the extreme value of $f(x, y, z) = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$. (10)
- (d) Find the residue of $\frac{z^3}{z^2 - 1}$ at the infinity. (10)
- (e) Show that the function $e^{\frac{1}{z^2}}$ has singularities. (10)
2. (a) Let $o(G) = 108$. Show that there exists a normal subgroup of order 27 or 9. (15)
- (b) Show that the sequence $\{\tan^{-1} nx\}, x \geq 0$ is uniformly convergent on any interval $[a, b], a > 0$ but is only pointwise convergent on $[0, b]$. (15)
- (c) Prove that $\int_0^\infty \frac{\sin mx}{x(x^2 + a^2)^2} dx = \frac{\pi}{2a^4} - \frac{\pi e^{-ma}}{4a^3} \left(m + \frac{2}{a}\right), m > 0, a > 0$. (20)
3. (a) Prove that $x^2 + x + 4$ is irreducible over F , the field of integer modulo 11 and prove further that $\frac{F[X]}{(x^2 + x + 4)}$ is a field having 121 elements. (20)
- (b) Prove Cauchy's Mean value theorem. (15)
- (c) Find the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$. (15)
4. (a) If (x, y, z) be the lengths of perpendiculars drawn from any interior point P of a triangle ABC on the sides BC, CA and AB respectively, then find the minimum value of $x^2 + y^2 + z^2$, the sides of the triangle ABC being a, b, c . (20)

- (b) An automobile dealer wishes to put four repairmen R_1, R_2, R_3 and R_4 to four different jobs J_1, J_2, J_3 and J_4 . But R_3 cannot do the job J_2 . The dealer has estimated the numbers of man hours that would be required for each job man on one-one basis as given in the following table:

	R_1	R_2	R_3	R_4
J_1	6	2	3	4
J_2	9	7	–	5
J_3	6	4	7	5
J_4	6	8	8	9

Formulate the above as Linear Programming Problem. (15)

- (c) Prove that a basic feasible solution to a linear programming problem must correspond to an extreme point of the set of all feasible solutions. (15)

SECTION-B

5. (a) Form a partial differential equation by eliminating the arbitrary functions $5S$ and $5T$ from $z = yf(x) + xg(y)$. (10)
- (b) Using Lagrange's interpolation formula, find a polynomial which passes the points $(0, -12), (1, 0), (3, 6), (4, 12)$. (10)
- (c) Draw a flow chart for solving equation $F(x) = 0$, correct to five decimal places by Newton-Raphson method. (10)
- (d) A heavy circular disk is revolving in a horizontal plane about the centre which is fixed. An insect of mass $\frac{1}{n}$ th that of the disk walks from the centre along a radius and flies away. Show that the final angular velocity is $\frac{n}{n+2}$ times the original angular velocity of the disk. (10)
- (e) Does a fluid with velocity $\vec{q} = \left[z - \frac{2x}{r}, 2y - 3z - \frac{2y}{r}, x - 3y - \frac{2z}{r} \right]$ possess vorticity, where $\vec{q}(u, v, w)$ is the velocity in the Cartesian frame $\vec{r}(x, y, z)$ and $r^2 = x^2 + y^2 + z^2$? What is the circulation in the circle $x^2 + y^2 = 9, z = 0$? (10)
6. (a) Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0, y = 0$. (20)

- (b) A function f , defined on $[0, 1]$, is such that $f(0) = 0$, $f\left(\frac{1}{2}\right) = -1$, $f(1) = 0$. Find the quadratic polynomial $p(x)$ which agrees with $f(x)$ for $x = 0, \frac{1}{2}, 1$. If

$$\left| \frac{d^3 f}{dx^3} \right| \leq 1 \text{ for } 0 \leq x \leq 1, \text{ show that } |f(x) - p(x)| \leq \frac{1}{12} \text{ for } 0 \leq x \leq 1. \quad (15)$$

- (c) Consider the integers 13, 8 and their 8 bit binary representations. Perform the following bitwise operations and express the results in decimal system:
- 13 & 20 (bitwise AND)
 - 13 | 20 (bitwise OR)
 - 13 ^ 20 (bitwise XOR)
 - ~ 20 (bitwise Complement)

(15)

7. (a) Solve the PDE $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y} - x^2y$. (15)

- (b) Evaluate $\int_0^1 e^{-x^2} dx$; employing three-point Gauss Quadrature formula, finding the required weights and residues. Use five decimal places for computation. (15)

- (c) A particle of unit mass is projected so that its total energy is h in a field of force of which the potential energy is $\phi(r)$ at a distance r from the origin. By employing the principle of energy and least action, show that the path is

$$\text{given by } c^2 \left[r^2 + \left(\frac{dr}{d\theta} \right)^2 \right] = r^4 [h - \phi(r)], \text{ where } c \text{ is constant.} \quad (20)$$

8. (a) Consider a uniform flow U_0 in the positive x -direction. A cylinder of radius a is located at the origin. Find the stream function and the velocity potential. Find also the stagnation points. (15)

- (b) The space between two infinitely long coaxial cylinders of radii a and b ($b > a$) respectively is filled by homogeneous fluid, of density ρ . The inner cylinder is suddenly moved with velocity v perpendicular to this axis, the outer being kept fixed. Show that the resulting impulsive pressure on a length l of inner

$$\text{cylinder is, } \pi \rho a^2 l \frac{b^2 + a^2}{b^2 - a^2} v. \quad (20)$$

- (c) Find the steady state temperature distribution in a thin rectangular plate bounded by the lines $x=0, x=a, y=0$ and $y=b$. The edges $x=0, x=a$ and $y=0$ are kept at temperature zero while the edge $y=b$ is kept at 100° . (15)



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(1a)

Let R be Ring of Gaussian Integers.

~~ie $R = \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$~~

$$R = \{a+bi \mid a, b \in \mathbb{Z}\}$$

clearly $1+i \in R$

Now R is also an Euclidean domain

$\Rightarrow$ (prime element $\Leftrightarrow$ irreducible element) in R .

1. we will show $1+i$ is irreducible

then $1+i = (a+ib)(c+id)$ where $a+ib, c+id \in R$

multiplying by conjugate

$$2 = (a^2+b^2)(c^2+d^2)$$

$\therefore a, b, c, d$ are Integer

$$\therefore a^2+b^2=2, \quad c^2+d^2=1 \quad \text{or}$$

$$a^2+b^2=1, \quad c^2+d^2=2$$

without loss of generality,

$$a^2+b^2=2 \quad \& \quad c^2+d^2=1$$

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$a^2 + b^2 = 2$ is not possible for any

integer values

hence, there cannot exist any such

gcd too.

$\therefore 1+i$ is irreducible in $\mathbb{Z}[i]$

$\Rightarrow \underline{1+i \text{ is prime element in } \mathbb{Z}[i]}$

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12

Given H is Subgroup of G .

$$\& \forall x \in G \quad x^2 \in H$$

To prove: H is Normal Subgroup

let any $h \in H$, a and $g \in G$

$$\text{then } (g h g^{-1})^2 \in H, \& g^{-1} \in H$$

$$\text{or } g h^2 g^{-1} \in H$$

$$\text{or } (g^{-1})^2 \cdot (g h^2 g^{-1}) \in H \quad (\text{Closure})$$

$$g^{-1} h^2 g^{-1} \in H$$

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1c

Given: $f(x, y, z) = 2x + 3y + z$

to find ~~max~~ extremum value of $f(x, y, z)$ with constraint $x^2 + y^2 = 5$ & $x + z = 1$

we will use Lagrange Multiplier Method

$$\text{let } F(x, y, z, \lambda, \mu) = f(x, y, z) + \lambda(x^2 + y^2 - 5) + \mu(x + z - 1)$$

Now, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$ will give stationary points

$$\frac{\partial F}{\partial x} = 2 + 2\lambda x + \mu = 0 \Rightarrow$$

$$\frac{\partial F}{\partial y} = 3 + 2\lambda y = 0 \Rightarrow \lambda = -\frac{3}{2y}$$

$$\frac{\partial F}{\partial z} = 1 + \mu = 0 \Rightarrow \mu = -1$$

Putting value of λ, μ in $\frac{\partial F}{\partial x} = 0$

$$2 + 2x\left(-\frac{3}{2y}\right) - 1 = 0$$

$$1 = \frac{3x}{y} \Rightarrow y = 3x$$

$$\text{i.e. } x^2 + (3x)^2 = 5 \Rightarrow x^2 = \frac{5}{10} = \frac{1}{2}$$

$$x = \pm \frac{1}{\sqrt{2}}$$

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$$\text{and also, } z = 1 - x = 1 \mp \frac{1}{\sqrt{2}}$$

$$\text{and } y = 3x = \pm \frac{3}{\sqrt{2}}$$

$\therefore \left(\pm \frac{1}{\sqrt{2}}, \pm \frac{3}{\sqrt{2}}, 1 \mp \frac{1}{\sqrt{2}} \right)$ will give extreme points values of $f(x, y, z)$

$$\begin{aligned} f\left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}\right) &= \frac{2}{\sqrt{2}} + \frac{9}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}} \\ &= 1 + 5\sqrt{2} \end{aligned}$$

$$\begin{aligned} f\left(-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}\right) &= -\frac{2}{\sqrt{2}} - \frac{9}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} \\ &= 1 - 5\sqrt{2} \end{aligned}$$

$\therefore$ Maximum Value of $f(x, y, z) = 1 + 5\sqrt{2}$

$$\text{@ } \left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}\right)$$

& Minimum Value of $f(x, y, z) = 1 - 5\sqrt{2}$

$$\text{@ } \left(-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}\right)$$

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(1d)

$$\text{let } f(z) = \frac{z^3}{z^2 - 1}$$

we need to find residue of f at $z \rightarrow \infty$.

We know that, $f(z)$ has singularity at $z=1$, $z=-1$ and $z \rightarrow \infty$.

Now, Since, sum of residue of all singularity is 0.

$$\text{i.e. } \text{residue}_f(z=1) + \text{res}_{f(z)}(z=-1) + \text{res}_{f(z)}(z=\infty) = 0$$

$$\therefore \text{Res}(z=\infty) = -[\text{Res}(z=1) + \text{Res}(z=-1)]$$

Residue of $f(z)$ at $z=1$

$f(z)$ has simple pole at $z=1$

$$\therefore \text{Res}(z=1) = \lim_{z \rightarrow 1} (z-1) f(z)$$

$$= \lim_{z \rightarrow 1} \frac{z^3}{z+1} = \frac{1}{2}$$

Also, Residue of $f(z)$ at $z=-1$

$f(z)$ has simple pole at $z=-1$

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$$\therefore \text{Res}(z=-1) = \lim_{z \rightarrow -1} (z+1)f(z)$$

$$= \lim_{z \rightarrow -1} \frac{z^3}{z-1} = \frac{-1}{-2} = \frac{1}{2}$$

$$\therefore \text{Res}(z=\infty) = - \left[\frac{1}{2} + \frac{1}{2} \right]$$

Residue at $z=\infty = -1$

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(2a)

let G be a group s.t. $|G| = 108$

To show, $\exists$ a Normal Subgroup of order 27/9.

let H be any Subgroup of G

then By Lagrange's theorem,

$$|H| \mid |G| \text{ i.e. } |H| \mid 108$$

$$\therefore |H| \nmid 2 \times 2 \times 3^3$$

We know that if any no. divides order of group, then of form p^a , where p is prime

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(2b)

$$\text{let } f_n(x) = \tan^{-1}(nx) \quad , x \geq 0$$

thus, sequence of function

$$\{f_n(x)\} = \{\tan^{-1}(nx)\} \text{ in } [a, b]$$

$$\text{let } f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \tan^{-1} nx = \frac{\pi}{2} \text{ as } \underline{x > 0}.$$

$$\begin{aligned} \text{Now Consider } |f_n(x) - f(x)| &= |\tan^{-1} nx - \pi/2| \\ &= |\tan^{-1}(y_n x)| \end{aligned}$$

$$\text{let } y = \tan^{-1} \frac{1}{nx} = \frac{\pi}{2} - \tan^{-1} nx$$

$$\frac{dy}{dx} = \frac{1}{1+x^2/n^2} \cdot \frac{-1 \times n}{1+n^2 x^2} = \frac{-n}{1+n^2 x^2} < 0$$

$\Rightarrow$ y is decreasing function.

$\therefore$ Maximum value of y will be at $\underline{x=a}$

$$\therefore \sup |f_n(x) - f(x)| = |\tan^{-1}(y_n a)| = M_n \text{ (say)}$$

$$\& \lim_{n \rightarrow \infty} M_n = \lim_{n \rightarrow \infty} \tan^{-1}(y_n a) = 0$$

$\therefore$ By M_n-Test, $\{f_n(x)\}$ is

uniformly Convergent on $[a, b]$

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Now, on interval $[0, b]$

① $\{f_n(x)\}$ is pointwise convergent for any $x \in (0, b]$.

$$f_n(x) = \tan^{-1} nx$$

$\therefore$ Sequence $\{ \tan^{-1} x, \tan^{-1} 2x, \dots, \tan^{-1} nx \}$

$$\text{and as } n \rightarrow \infty \lim_{n \rightarrow \infty} \tan^{-1} nx = \pi/2$$

$\therefore$ pointwise convergent for $x = x \in (0, b]$

Now for $x = 0$

Sequence of function will be

$$\begin{aligned} \{f_n(0)\} &= \{ \tan^{-1}(0), \tan^{-1}(0), \dots \} \\ &= \{ 0, 0, \dots \} \end{aligned}$$

$\therefore$ All values are 0

$\therefore$ pointwise convergent at $x = 0$

② Uniform Convergence

$$\text{Consider } f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

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$$f(x) = \begin{cases} 0 & x=0 \\ \pi/2 & x \neq 0, x \in (0, b] \end{cases}$$

$\therefore$ limit function is discontinuous.

& we know, if $\{f_n(x)\}$ cgs uniformly

then $\lim_{n \rightarrow \infty} f_n(x)$ is continuous as

$\{f_n(x)\}$ is also continuous

$\therefore \{f_n(x)\}$ cannot be uniformly convergent on $[0, b]$

$\therefore \{ \tan^{-1}(nx) \}$ does not converges uniformly but pointwise on $[0, b]$

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2c

To prove:
$$\int_0^{\infty} \frac{\sin mx}{x(x^2+a^2)^2} dx = \frac{\pi}{2a^4} - \frac{\pi e^{-ma}}{4a^3} \left(m + \frac{2}{a}\right)$$

 $m > 0, a > 0$

Let $I_1 = \int_0^{\infty} \frac{\sin mx}{x(x^2+a^2)^2} dx = \text{Imaginary part of } \int_0^{\infty} \frac{e^{imx}}{x(x^2+a^2)^2} dx$

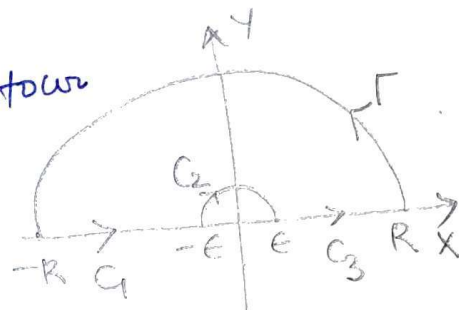
Now, let $f(z) = \frac{e^{imz}}{z(z^2+a^2)^2}$

& $I = \int_0^{\infty} \frac{e^{imx}}{x(x^2+a^2)^2} dx$

We will construct a Contour for Contour Integration

$C = \Gamma + C_1 + C_2 + C_3$

s.t. $\Gamma: z(t) = Re^{it} \quad t=0 \text{ to } \pi$
 $C_1: z(t) = t \quad t=-R \text{ to } -\epsilon$
 $C_2: z(t) = \epsilon e^{it} \quad t=\pi \text{ to } t=0$
 $C_3: z(t) = t \quad t=\epsilon \text{ to } R.$



Now, Singular points of $f(z)$ are
 $z=0, z=ai$ & $z=-ai$

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and only $z = ai$ lie inside C .

$\therefore f(z)$ is analytic on C and inside C except at $z = ai$.

Now, we can write $I = \oint_C f(z) dz$

And, By Residue theorem

$$\oint_C f(z) dz = 2\pi i (\text{Residue of } f \text{ at } z = ai)$$

$$(\text{Residue of } f \text{ at } z = ai) = \lim_{z \rightarrow ai} \frac{d}{dz} ((z - ai)^2 f(z))$$

$$= \lim_{z \rightarrow ai} \frac{d}{dz} \left(\frac{e^{imz}}{z(z+ai)^2} \right)$$

$$= \lim_{z \rightarrow ai} \frac{(ime^{imz}(z)(z+ai)^2 - 2(z+ai)(z)e^{imz} - (z+ai)^2 e^{imz})}{z^2(z+ai)^4}$$

$$= \lim_{z \rightarrow ai} \frac{e^{imz} [z+ai] [imz(z+ai) - 2z(z+ai)]}{z^2(z+ai)^4}$$

$$= \lim_{z \rightarrow ai} \frac{e^{imz}}{z^2(z+ai)^3} [imz(z+ai) - 2z(z+ai)]$$

$$= \frac{e^{-ma}}{-a^2(2ai)^3} [-ma(2ai) - 2ai - 2ai]$$

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$$= \frac{e^{-ma}}{-a^2 (2ai)^3} (2ai) (-ma - 2)$$

$$= \frac{e^{-ma}}{a^2 (2ai)^2} (ma + 2)$$

$$\text{Res} \approx \boxed{\frac{e^{-ma}}{-4a^3} \left(m + \frac{2}{a}\right)}$$

$$\text{Now, } I = \oint_C f(z) dz = \oint \frac{2\pi i e^{-ma}}{-4a^3} \left(m + \frac{2}{a}\right)$$

$$\text{Now, } \oint_C f(z) dz = \oint_{\Gamma} f(z) dz + \oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz + \oint_{C_3} f(z) dz$$

$$\oint_{\Gamma} f(z) dz = \oint_{\Gamma} \frac{e^{imz}}{z(z^2 + a^2)^2} dz$$

$$\text{Consider } g(z) = \frac{1}{z(z^2 + a^2)^2}$$

$$\text{And as } z \rightarrow \infty, \lim_{z \rightarrow \infty} g(z) = 0$$

$\therefore$ By Jordan's lemma,

$$\boxed{\lim_{R \rightarrow \infty} \oint_{\Gamma} e^{imz} g(z) dz = 0}$$

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Consider, $\oint_{\Gamma_2} f(z) dz = \oint_{\Gamma_2} \frac{e^{imz}}{z(z^2+a^2)^2} dz$

$$= \int_{-\pi}^0 \frac{e^{im\epsilon e^{it}} \cdot i\epsilon e^{it} dt}{\epsilon e^{it} (\epsilon^2 e^{i2t} + a^2)^2}$$

$$= \int_{-\pi}^0 \frac{e^{im\epsilon \cos t} \cdot e^{-m\epsilon \sin t} \cdot i dt}{(a^2 + \epsilon^2 e^{i2t})^2}$$

and as $\epsilon \rightarrow 0$

$$\oint_{\Gamma_2} f(z) dz = \int_{-\pi}^0 \frac{i dt}{a^4} = -\frac{i\pi}{a^4}$$

Now $\oint_C f(z) dz = \frac{2\pi i e^{-ma}}{-4a^3} \left(m + \frac{2}{a}\right)$

taking $\epsilon \rightarrow 0$ & $R \rightarrow \infty$

$$\oint_{\Gamma_1} f(z) dz + \oint_{\Gamma_3} f(z) dz = \frac{i\pi}{a^4} - \frac{2\pi i e^{-ma}}{4a^3} \left(m + \frac{2}{a}\right)$$

$$2 \int_{\epsilon}^R f(z) dz = 2 \int_0^{\infty} f(z) dz = \frac{i\pi}{a^4} - \frac{2\pi i e^{-ma}}{4a^3} \left(m + \frac{2}{a}\right)$$

$$\Rightarrow \int_0^{\infty} \frac{e^{imx}}{x(x^2+a^2)^2} dx = \frac{i\pi}{2a^4} - \frac{i\pi e^{-ma}}{4a^3} \left(m + \frac{2}{a}\right)$$

Comparing Imaginary part $\rightarrow$

$$\int_0^{\infty} \frac{\sin mx}{x(x^2+a^2)^2} dx = \frac{\pi}{2a^4} - \frac{\pi e^{-ma}}{4a^3} \left(m + \frac{2}{a}\right)$$

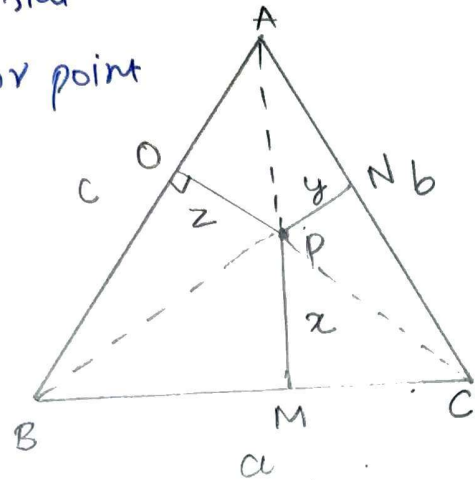
4a

Let ABC be triangle with side a, b and c . Let P be interior point with $\perp$ distance from

BC as $PM = x$

AC as $PN = y$

AB as $PO = z$



Now, we need to find minimum value of $x^2 + y^2 + z^2$

We know, ABC has fixed lengths as a, b, c

$$\therefore \text{Area of } \triangle ABC = s_0 = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{--- (1)}$$

where $s = \frac{a+b+c}{3}$

$$\& \text{ also } \text{Ar}(\triangle ABC) = \text{Area}(APC) + \text{Area}(BPC) + \text{Area}(APB)$$

$$\& \text{ Area of } APB = \frac{1}{2} c \times z = \frac{cz}{2}$$

$$\text{Area of } BPC = \frac{1}{2} a \times x = \frac{ax}{2}$$

$$\text{Area of } APC = \frac{by}{2}$$

$$\therefore s_0 = \frac{ax + by + cz}{2}$$

$$\text{or at } 2s_0 = k \text{ then } ax + by + cz = k \quad \text{--- (2)}$$

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Now, we need to maximize $x^2 + y^2 + z^2$
with constraint $ax + by + cz = k$
we will use Lagrange Multiplier Method

$$\text{let } F(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(ax + by + cz - k)$$

Now, for stationary point $\rightarrow$

$$F_x = F_y = F_z = 0$$

$$\Rightarrow 2x + \lambda a = 0 \Rightarrow x = -\frac{\lambda a}{2}$$

$$2y + \lambda b = 0 \Rightarrow y = -\frac{\lambda b}{2}$$

$$2z + \lambda c = 0 \Rightarrow z = -\frac{\lambda c}{2}$$

Substituting x, y, z in eq (2)

$$-\frac{\lambda}{2}(a^2 + b^2 + c^2) = k$$

$$\Rightarrow \lambda = -\frac{2k}{a^2 + b^2 + c^2}$$

$$\Rightarrow x = \frac{ka}{a^2 + b^2 + c^2}$$

$$\& y = \frac{kb}{a^2 + b^2 + c^2} \quad \& z = \frac{kc}{a^2 + b^2 + c^2}$$

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$$\begin{aligned} \therefore k^2 + y^2 + z^2 &= \frac{k^2(a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} \\ &= \frac{k^2}{a^2 + b^2 + c^2} \\ &= \frac{4s_0^2}{a^2 + b^2 + c^2}, \text{ where } s_0 \text{ is area of } \triangle ABC \end{aligned}$$

$\therefore$ Now, to check minimum

$$d^2F = \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_{x\lambda} \\ F_{yx} & F_{yy} & F_{yz} & F_{y\lambda} \\ F_{zx} & F_{zy} & F_{zz} & F_{z\lambda} \\ F_{\lambda x} & F_{\lambda y} & F_{\lambda z} & F_{\lambda\lambda} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 & 0 & a \\ 0 & 2 & 0 & b \\ 0 & 0 & 2 & c \\ a & b & c & 0 \end{vmatrix} = 2 \begin{vmatrix} 2 & 0 & b \\ 0 & 2 & c \\ b & c & 0 \end{vmatrix} - a \begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ a & b & c \end{vmatrix}$$

$$= 2(2(-c^2) + b(-2b)) - a(a \cdot 4)$$

$$= -4(a^2 + b^2 + c^2) < 0$$

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$$\therefore d^2 F < 0 \quad \text{at } x = \frac{ka}{a^2+b^2+c^2}, y = \frac{kb}{a^2+b^2+c^2}$$

$$z = \frac{kc}{a^2+b^2+c^2}$$

$\therefore$ Value of $x^2+y^2+z^2$ is Minimum

$\therefore$ Minimum Value of $x^2+y^2+z^2$

$$= \frac{4s^2}{a^2+b^2+c^2}$$

$$= \frac{4s(s-a)(s-b)(s-c)}{a^2+b^2+c^2}, \text{ where } s = \frac{a+b+c}{3}$$

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4b

Given problem is Assignment problem with 4 repairmen (R_1, R_2, R_3, R_4) & Jobs (J_1, J_2, J_3, J_4) with following hours $\rightarrow$

let x_{ij} be a binary variable which represent

	R_1	R_2	R_3	R_4
J_1	6	2	3	4
J_2	9	7	-	5
J_3	6	4	7	5
J_4	6	8	8	9

$x_{ij} = 1$ if i th Job assigned to j th Repairman.

$x_{ij} = 0$ otherwise
& $i = 1, 2, 3, 4$ & $j = 1, 2, 3, 4$

$\therefore$ Job 2 cannot be assigned to R_3 .

$$\therefore \boxed{x_{23} = 0}$$

Now, we will write conditions. —

each job must be done by some Repairmen: $\therefore x_{11} + x_{12} + x_{13} + x_{14} = 1$

$$x_{21} + x_{22} + x_{24} = 1$$

$$(\because x_{23} = 0)$$

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$$x_{31} + x_{32} + x_{33} + x_{34} = 1$$

$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$

similarly every repairman has to do one job

$$\therefore x_{11} + x_{21} + x_{31} + x_{41} = 1$$

~~$$x_{21} + x_{22} + x_{23} +$$~~

$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$

$$x_{13} + x_{33} + x_{43} = 1$$

$$\therefore x_{23} = 0$$

$$\text{and } x_{14} + x_{24} + x_{34} + x_{44} = 1$$

Also, $x_{ij} = 0$ or 1 only

Now, Cost of Assignment will be

Minimize,

$$Z = 6x_{11} + 2x_{12} + 3x_{13} + 4x_{14} + 5x_{24} + 9x_{21} + 7x_{22} + 6x_{31} + 4x_{32} + 7x_{33} + 5x_{34} + 6x_{41} + 8x_{42} + 8x_{43} + 9x_{44}$$

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with Constraint $\rightarrow x_{23} = 0, x_{ij} = 0 \text{ or } 1$

$$x_{11} + x_{12} + x_{13} + x_{14} = 1$$

$$x_{21} + x_{22} + x_{24} = 1$$

$$x_{31} + x_{32} + x_{33} + x_{34} = 1$$

$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$

$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$

$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$

$$x_{13} + x_{33} + x_{43} = 1$$

$$x_{14} + x_{24} + x_{34} + x_{44} = 1$$

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5a

Given function : $z = y f(x) + x g(y)$ — (i)

we need to eliminate f and g and form p.d.e.

differentiating eq (i) w.r.t. x & y →

$$p = \frac{\partial z}{\partial x} = y f'(x) + g(y) \quad \text{--- (ii)}$$

$$q = \frac{\partial z}{\partial y} = f(x) + x g'(y) \quad \text{--- (iii)}$$

differentiating again w.r.t. x & y

$$r = \frac{\partial^2 z}{\partial x^2} = y f''(x)$$

$$s = \frac{\partial^2 z}{\partial y \partial x} = f'(x) + g'(y) \quad \text{--- (iv)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = x g''(y)$$

Now, multiplying eq (i) with x & (ii) with y & adding →

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$$px + qy = xy (f'(y) + g'(y)) + xg(y) + yf(x)$$

Using eq ① & ②

$$px + qy = xy \left(\frac{\partial^2 z}{\partial x \partial y} \right) + z$$

$\therefore$ partial differential eqⁿ for eq ① is $\rightarrow$

$$\boxed{(xy) \frac{\partial^2 z}{\partial x \partial y} - px - qy + z = 0}$$

UPSC

5b

Given points
are $\rightarrow$

$i \rightarrow$	0	1	2	3
x_i	0	1	3	4
y_i	-12	0	6	12

Let $\Pi_4(x)$ be a polynomial s.t

$$\Pi_4(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

then, By Lagrange's Interpolation Formula

$$P_3(x) = \sum_{i=0}^3 \frac{\Pi_4(x)}{(x-x_i) \Pi_4'(x_i)} \times y_i$$

$$= \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} (-12) + 0$$

$$+ \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} 6 + \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} (12)$$

$$= \frac{(x-1)(x^2-7x+12)}{(-12)} (-12) + \frac{x(x^2-4x+3)}{-2} (6)$$

$$+ \frac{x(x^2-3x+2)}{6} (12)$$

$$= (x^3 - 8x^2 + 19x - 12) - 3(x^3 - 4x^2 + 3x) + 2(x^3 - 3x^2 + 2x)$$

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$$P_3(x) = x^3(0) + x^2(-2) + 14x - 12$$

$$\therefore \boxed{P_3(x) = -2x^2 + 14x - 12}$$

is required polynomial from
Lagrange's Interpolation formula

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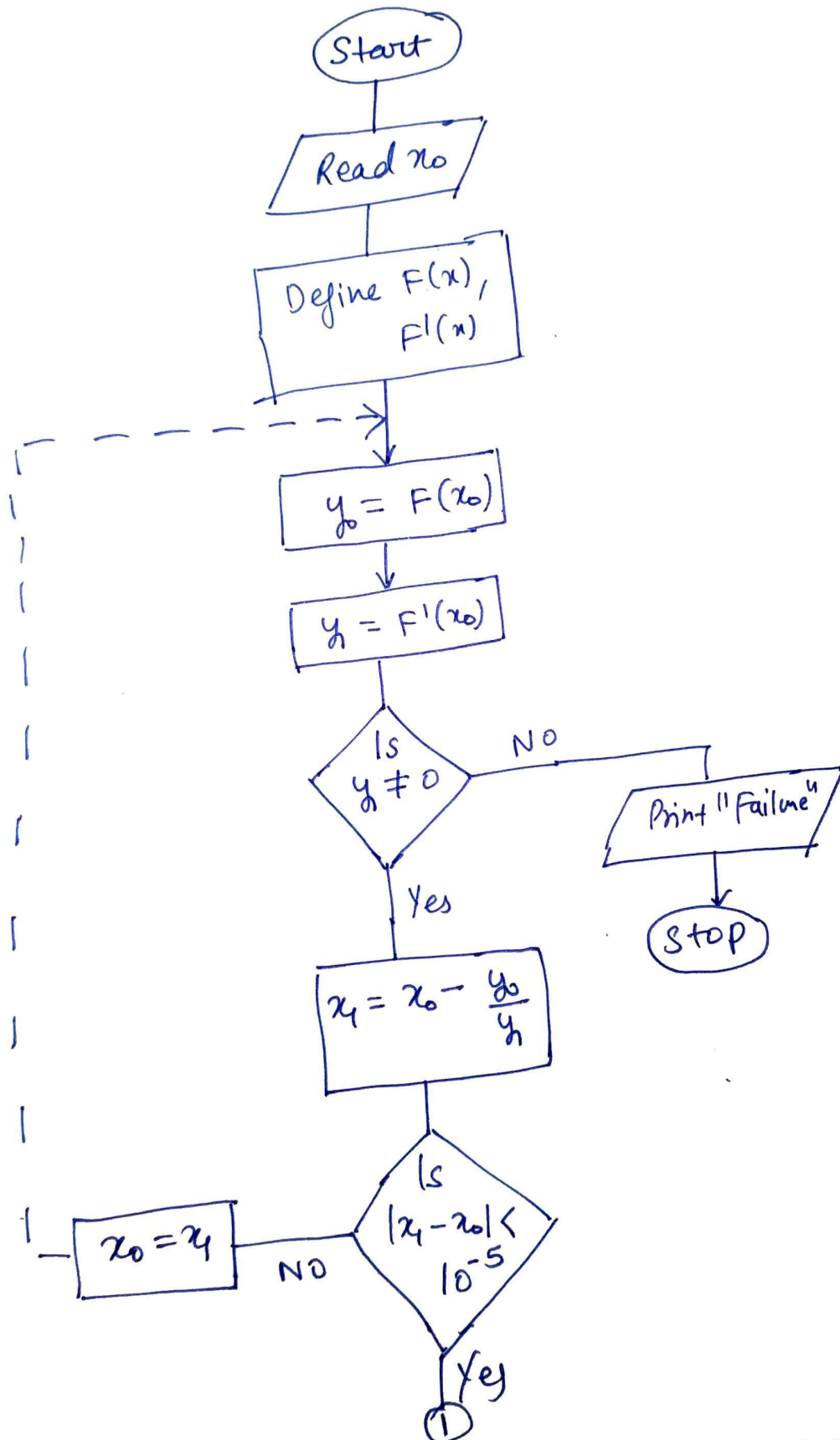
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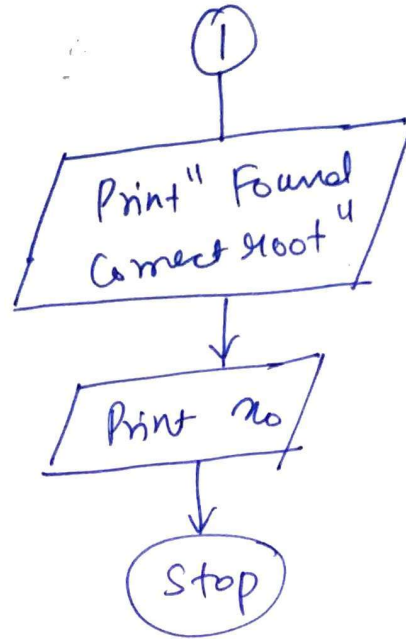
Given: function : $F(x) = 0$



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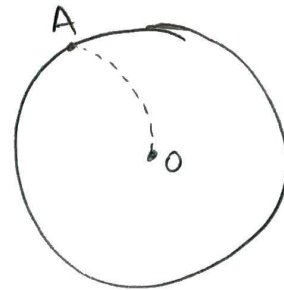
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5d

Let Circular disk with Centre fixed having mass $\frac{nM}{2}$ and fly of mass M rotate about its axis with Angular velocity ω



Disk with motion of fly

Now, As fly move from Centre O to boundary A there are only internal forces on the system & thus angular momentum will be conserved.

$$\begin{aligned} \text{Initial Angular momentum} &= I\omega \\ &= \left(\frac{nMR^2}{2} + M(0) \right) \omega = \frac{nMR^2\omega}{2} \end{aligned}$$

$\therefore$ fly is at Centre.

Now final Angular momentum will be $\rightarrow$

$$= \left(\frac{nMR^2}{2} + MR^2 \right) \omega', \text{ where } \omega' \text{ is new angular velocity}$$

& Both will be same

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$$\therefore \frac{nMR^2}{2} \omega = \left(\frac{nMR^2}{2} + MR^2 \right) \omega'$$

$$\omega' = \frac{n}{n+2} \omega$$

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(8e)

given fluid velocity:

$$\vec{q} = \left[z - \frac{2x}{r}, 2y - 3z - \frac{2y}{r}, x - 3y - \frac{2z}{r} \right]$$

$$\text{where } r^2 = x^2 + y^2 + z^2$$

we know Vorticity $\Omega = \nabla \times \vec{q}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - \frac{2x}{r} & 2y - 3z - \frac{2y}{r} & x - 3y - \frac{2z}{r} \end{vmatrix}$$

$$= \hat{i} \left(-3 - 2z \cdot \left(-\frac{1}{r^2}\right) \frac{\partial r}{\partial y} + 3 + 2y \left(-\frac{1}{r^2}\right) \left(\frac{\partial r}{\partial z}\right) \right)$$

$$+ \hat{j} \left(1 - 2z \left(-\frac{1}{r^2}\right) \frac{\partial r}{\partial x} - 1 + 2x \left(-\frac{1}{r^2}\right) \frac{\partial r}{\partial z} \right)$$

$$+ \hat{k} \left(-2y \left(-\frac{1}{r^2}\right) \left(\frac{\partial r}{\partial x}\right) + 2x \left(-\frac{1}{r^2}\right) \frac{\partial r}{\partial y} \right)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\therefore \text{Vorticity} = \Omega = \nabla \times \vec{q} = 0.$$

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6a

Let given p.d.e be

$$f(x, y, z, p, q) = p^2 - q^2 - z = 0. \quad \text{--- (I)}$$

and given that Integral surface passes through
parabola $4z + x^2 = 0, y = 0$

writing parabola in parametric form $\rightarrow$

$$x_0 = 2\lambda \quad y_0 = 0 \quad z_0 = -\lambda^2 \quad \text{--- (II)}$$

Now, characteristic equations of p.d.e (I) is

$$x'(t) = f_p \quad y'(t) = f_q \quad z'(t) = pf_p + qf_q$$

$$p'(t) = -f_x - pf_z \quad q'(t) = -f_y - qf_z$$

putting values from eq (I)

$$x'(t) = 2p \quad y'(t) = -2q \quad z'(t) = 2p^2 - 2q^2$$

$$p'(t) = -0 - p(-1) = p \quad q'(t) = 0 - q(-1) = q$$

$$\Rightarrow p'(t) = p \quad \text{and} \quad q'(t) = q. \quad \text{--- (III)}$$

Now, we will calculate initial values of p, q

If we take $x_0 = f_1(\lambda) = 2\lambda \quad y_0 = f_2(\lambda) = 0$

$$z_0 = f_3(\lambda) = -\lambda^2$$

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$$\text{then, } f_3'(\lambda) = f_1'(\lambda) p_0 + f_2'(\lambda) q_0$$

$$-2\lambda = 2p_0 + 0 \Rightarrow \boxed{p_0 = -\lambda}$$

Now, from eq ① we will calculate q_0

$$p_0^2 - q_0^2 - z_0 = 0$$

$$\lambda^2 - q_0^2 + \lambda^2 = 0 \Rightarrow \boxed{q_0 = \sqrt{2}\lambda}$$

$$\therefore \text{Initial Values } x_0 = 2\lambda, y_0 = 0, z_0 = -\lambda^2$$

$$p_0 = -\lambda, q_0 = \sqrt{2}\lambda$$

Using these to find solution of characteristic eqⁿ

$$\text{Consider, } p'(t) = p \quad \& \quad q'(t) = q$$

$$\frac{dp}{dt} = p$$

$$\Rightarrow \log p = t + c$$

$$\Rightarrow p = ce^t$$

$$\text{at } t=0, p_0 = -\lambda$$

$$\Rightarrow \boxed{p = -\lambda e^t}$$

$$\frac{dq}{dt} = q$$

$$\log q = t + c$$

$$q = c'e^t$$

$$\text{at } t=0, q_0 = \sqrt{2}\lambda$$

$$\Rightarrow \boxed{q = \sqrt{2}\lambda e^t}$$

$$\text{Now, } x'(t) = 2p'(t)$$

$$\frac{dx}{dt} = 2(-\lambda e^t)$$

$$\Rightarrow x = -2\lambda e^t + C''$$

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$$\text{at } t=0 \quad x_0 = 2\lambda \Rightarrow c'' = 4\lambda$$

$$\boxed{x = 2\lambda(2 - e^t)}$$

$$\text{Also, } \frac{dy}{dt} = -2(\sqrt{2}\lambda e^t)$$

$$y = -2\sqrt{2}\lambda e^t + c_3 \quad \text{at } t=0, y_0 = 0$$

$$\Rightarrow c_3 = 2\sqrt{2}\lambda \Rightarrow \boxed{y = 2\sqrt{2}\lambda(1 - e^t)}$$

$$\& \quad z = p^2 - q^2 = e^{2t}(\lambda^2 - 2\lambda^2)$$

$$\boxed{z = -\lambda^2 e^{2t}}$$

Now, we have to eliminate λ, t .

$$x = 4\lambda - 2\lambda e^t \quad y = 2\sqrt{2}\lambda - 2\sqrt{2}\lambda e^t$$

multiplying y by $\sqrt{2}$ & subtracting

$$x - \sqrt{2}y = 4\lambda - 4\lambda - 2\lambda e^t + 4\lambda e^t$$

$$x - \sqrt{2}y = 2\lambda e^t$$

$$(x - \sqrt{2}y)^2 = 4\lambda^2 e^{2t} = 4(-z)$$

$$\boxed{4z + (x - \sqrt{2}y)^2 = 0}$$

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or

$$x^2 + 2y^2 - 2\sqrt{2}xy + 4z = 0$$

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(6b) Let $f: [0, 1] \rightarrow \mathbb{R}$ be function s.t

$$f(0) = 0, f\left(\frac{1}{2}\right) = -1, f(1) = 0$$

x	0	$\frac{1}{2}$	1
$f(x)$	0	-1	0

We have to find

quadratic polynomial $p(x)$ agreeing to above values.

Let $p(x) = ax^2 + bx + c$ which satisfy

value of x in above Table.

$$\therefore p(0) = 0 \Rightarrow c = 0$$

$$p(1) = 0 \Rightarrow a + b = 0 \Rightarrow b = -a$$

$$p\left(\frac{1}{2}\right) = -1 \Rightarrow \frac{a}{4} + \frac{b}{2} = -1$$

$$\frac{a}{4} - \frac{a}{2} = -1$$

$$\Rightarrow -\frac{a}{4} = -1 \Rightarrow \boxed{a = 4}$$

$$\& \boxed{b = -4}$$

$$\therefore \boxed{p(x) = 4x^2 - 4x}$$

is required quadratic polynomial satisfying above values.

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6c

8 bit binary representation of 13, 20

2	8	13	1
2	6	0	
2	3	1	
	1		

$$(13)_{10} = (00001101)_2$$

2	20	0
2	10	0
2	5	1
2	2	1
	1	

$$(20)_{10} = (00011100)_2$$

(i) Bitwise AND -

$$(13)_{10} = (00001101)_2$$

$$(20)_{10} = (00011100)_2$$

$$13 \& 20 = (00001100)_2$$

$$13 \& 20 = (12)_{10}$$

(ii) Bitwise OR →

$$(13)_{10} = (00001101)_2$$

$$(20)_{10} = (00011100)_2$$

$$(13 | 20)_{10} = (00011101)_2$$

16 8 4 2 1 2

$$= (29)_{10}$$

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$$\boxed{(13 \mid 20)_{10} = (29)_{10}}$$

(iii) bitwise XOR

$$(13)_{10} = (0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1)_2$$

$$(20)_{10} = (0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0)_2$$

$$(13 \wedge 20)_{10} = (0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1)_2$$

$$\boxed{(13 \wedge 20)_{10} = (17)_{10}}$$

(iv) bitwise complement of 20

$$(20)_{10} = (0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0)_2$$

$$(\sim 20)_{10} = (\overset{7}{1} \ \overset{6}{1} \ \overset{5}{1} \ \overset{4}{0} \ \overset{3}{0} \ \overset{2}{0} \ \overset{1}{1} \ \overset{0}{1})_2$$

$$\boxed{(\sim 20)_{10} = (227)_2}$$