

NEXT IAS

(To be filled by candidate)

Name of Candidate: Kamishak Aggarwal

Roll No.: MT 24MATL1011 Date of Examination: 31/7

Mobile Number: 7milom E-mail: 31/7

Exam Centre: Old Rajinder Nagar ☐ Online ☒

CSE (MAINS) TEST SERIES-2024

Test ~~001~~ <u>12</u>	MATHEMATICS
Dated : 14-07-2024	OPTIONAL

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are EIGHT questions divided in Two sections and printed in ENGLISH.

Candidate has to attempt FIVE questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question cum Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

DELHI CENTRE:

Old Rajinder Nagar: 27-B, Pusa Road, Metro Pillar no. 118, Near Karol Bagh Metro, New Delhi-110060 | Phone: 8081300200.

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BHOPAL CENTRE: Plot No. 46 Zone - 2, M.P. Nagar, Bhopal - 462011 | Phone: 8827664812, 8081300200.**JAIPUR CENTRE:** Plot No. 6 & 7, 3rd Floor, Stee Gopal Nagar, Gopalpura Bypass, Jaipur - 302015 | Phone: 9358200511.**PRAYAGRAJ CENTRE:** 31/31 Sardar Patel Marg, Civil Lines, Prayagraj, Uttar Pradesh - 211001 | Phone: 9058857757.

(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	15			
2. (c)	20	20			
3. (a)	24	20			
3. (b)	28	15			
3. (c)	31	15			
4. (a)	34	20			
4. (b)	38	15			
4. (c)	41	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	15			
7. (c)	70	20			
8. (a)	74	15			
8. (b)	77	20			
8. (c)	81	15			
Grand Total					

Remarks

Observations:

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{pmatrix}$$

$$W = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{pmatrix}$$

SECTION-A

1. (a) Show that $(1+i)$ is prime element in the ring R of Gaussian integers. (10)

$$R = \{a+bi \mid a, b \in \mathbb{Z}\}$$

$$(1+i) = (a+bi)(c+di)$$

$$(1-i) = (a-bi)(c-di)$$

$$1+1 = (a^2+b^2)(c^2+d^2) = 2 \cdot 1$$

$$a^2+b^2=2$$

$$c^2+d^2=1$$

$$a^2+b^2=1$$

$$c^2+d^2=2$$

~~$$a=b$$~~

$$a = \pm 1$$

$$b = \pm 1$$

$$c = \pm 1$$

$$d = 0$$

$$a = \pm 1$$

$$b = \pm 1$$

$$c = 0$$

$$d = \pm 1$$

$$a = \pm 1$$

$$b = 0$$

$$c = \pm 1$$

$$d = \pm 1$$

$$a = 0$$

$$b = \pm 1$$

$$c = \pm 1$$

$$d = \pm 1$$

~~$$(1+i) = (1-i)$$~~

~~$$c+d$$~~

~~$$c+d$$~~

$$(1+i), (-1-i), (1-i), (-1+i)$$

$$1$$

$$-1$$

$$i$$

$$-i$$

Hence irreducible element.

Q. let $(1+i)$ is ^{not} ~~not~~ prime

then $(1+i) \mid (a+bi)(c+di)$

$$(1-i) \mid (a-bi)(c-di)$$

$$2 \mid (a^2+b^2)(c^2+d^2)$$

$$\Rightarrow 2 \mid (a^2+b^2) \text{ or } 2 \mid c^2+d^2.$$

a, b both even or both odd.

if a, b both even

$$\text{then } 2 \mid a, 2 \mid b$$

$$\Rightarrow 2 \mid a+bi$$

$$\& (1+i) \mid 2 \Rightarrow (1+i) \mid a+bi$$

if ^{but} a, b both odd.

$$a+bi = 2(n+mi) + (1+i)$$

$$a = 2n+1$$

$$b = 2m+1$$

$$\Rightarrow (1+i) \mid a+bi \text{ as } (1+i) \mid 2 \& (1+i) \mid (1+i)$$

$\therefore (1+i)$ is prime element.

1. (b)

If H is a subgroup of a group G such that $x^2 \in H$ for every $x \in G$, then prove that H is a normal subgroup of G . (10)

To prove $ghg^{-1} \in H$ where $h \in H$

$$ghg^{-1} = ghgg^{-1}g^{-1}$$

$$= ghghh^{-1}g^{-1}g^{-1}$$

$$~~= (gh)(gh)(h^{-1}g^{-1})h^{-1}~~$$

$$= (gh)^2 h^{-1} (g^{-1})^2$$

$$h \in H \subset G, \quad g \in G, \quad g^{-1} \in G$$

$$\Rightarrow \cancel{gh} \in G$$

$$\Rightarrow gh \in G$$

$$\Rightarrow (g^{-1})^2 \in H$$

$$g \in G$$

$$\Rightarrow (gh)^2 \in H$$

$$h^{-1} \in H \quad (\text{as } H \leq G)$$

$$\Rightarrow (gh)^2 h^{-1} (g^{-1})^2 \in H$$

hence $ghg^{-1} \in H \quad \forall g \in G.$

1. (c)

Find the extreme value of $f(x, y, z) = 2x + 3y + z$ subject to $x^2 + y^2 = 5$ and $x + z = 1$. (10)

$$F = 2x + 3y + z + \lambda_1(x^2 + y^2 - 5) + \lambda_2(x + z - 1)$$

$$F_x = 2 + 2\lambda_1 x + \lambda_2 = 0$$

$$F_y = 3 + 2\lambda_1 y = 0$$

$$F_z = 1 + 0 + \lambda_2 = 0 \Rightarrow \lambda_2 = -1$$

$$xF_x + yF_y + zF_z = 0$$

$$(2x + 3y + z) + 2\lambda_1(x^2 + y^2) + \lambda_2(x + z) = 0$$

$$(2x + 3y + z) + 10\lambda_1 + \lambda_2 = 0$$

$$2 + 2\lambda_1 x - 1 = 0$$

$$\Rightarrow \lambda_1 x = -\frac{1}{2}$$

$$\lambda_1 y = -\frac{3}{2}$$

$$x^2 + y^2 = 5$$

$$\frac{1}{4\lambda_1^2} + \frac{9}{4\lambda_1^2} = 5$$

$$\lambda_1^2 = \frac{2}{4} \Rightarrow \lambda_1 = \pm \frac{1}{\sqrt{2}}$$

$$2x = -1/2$$

$$\Rightarrow x = \mp (\sqrt{2})^{-1}$$

$$2y = -3/2 \Rightarrow y = \mp 3(\sqrt{2})^{-1}$$

$$z = 1 - x = 1 \pm (\sqrt{2})^{-1}$$

$$f = 2x + 3y + z$$

$$= 2(\mp (\sqrt{2})^{-1}) \mp 9(\sqrt{2})^{-1} + 1 \pm (\sqrt{2})^{-1}$$

$$f_{\text{extremum}} = \frac{11\sqrt{2}}{2} - 10(\sqrt{2})^{-1} + 1, 10(\sqrt{2})^{-1} + 1$$

1. (d)

Find the residue of $\frac{z^3}{z^2-1}$ at the infinity.

$$f(z) = \frac{z^3}{z^2-1}$$

$$= \frac{z^3}{z^2(1 - 1/z^2)}$$

$$= z \left[1 - \frac{1}{z^2} \right]^{-1}$$

$$= z \left[1 + \frac{1}{z^2} + \frac{1}{z^4} + \dots \right]$$

$$= z + \frac{1}{z} + \frac{1}{z^3} + \dots$$

Negative coefficient of $\frac{1}{z}$ in expansion of $f(z)$ around infinity is (-1)

$$\text{Residue} = -1$$

1. (e)

Show that the function e^{-1/z^2} has singularities.

(10)

$$f(z) = e^{-1/z^2}$$

$$= \frac{1}{e^{1/z^2}}$$

$$z = x + iy$$

$$= \frac{1}{e^{\frac{1}{(x+iy)^2}}} = \frac{1}{e^{\frac{(x-iy)^2}{(x^2+y^2)^2}}}$$

$$= e^{-\left(\frac{x^2-y^2}{(x^2+y^2)^2} - \frac{2i xy}{(x^2+y^2)^2}\right)}$$

$$= e^{-\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)} e^{i\left(\frac{2xy}{(x^2+y^2)^2}\right)}$$

$$= e^{+\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)} \left[\cos \frac{2xy}{(x^2+y^2)^2} + i \sin \frac{2xy}{(x^2+y^2)^2} \right]$$

zeros/poles given by

$$e^{+\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)} = 0$$

$$\frac{x^2-y^2}{(x^2+y^2)^2} \rightarrow \infty$$

as along $y = mx$

as $(x, y) \rightarrow (0, 0)$

limit point of zeroes is

non isolated essential singularity
hence $(0+io)$ is singularity



3. (a)

Prove that $x^2 + x + 4$ is irreducible over F , the field of integer modulo 11 andprove further that $\frac{F[X]}{(x^2 + x + 4)}$ is a field having 121 elements. (20)

$$\text{let } \alpha \in \mathbb{Z}_{11}$$

$$\alpha^2 + \alpha + 4 = 0$$

No such α exists

~~$$\alpha^2 + \alpha + 4 = 0$$~~

α	$f(\alpha)$
0	4
1	6
2	10
3	5
4	2
5	1
6	2
7	5
8	10
9	6
10	4

$\therefore x^2 + x + 4$ is irreducible

$\Rightarrow x^2 + x + 4$ is irreducible element of $F[x]$
 $F[x]$ is BPID $\therefore \langle x^2 + x + 4 \rangle$ is maximal
 ideal of $F[x]$

$\therefore \frac{F[x]}{\langle x^2 + x + 4 \rangle}$ is a field

element of $\frac{F[x]}{(x^2+x+4)} = (x^2+x+4) + (ax+b)$

a has 11 choices

b has 11 choices.

$\therefore ax+b$ can be 121 combinations



3. (b)

Prove Cauchy's Mean Value Theorem

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Cauchy Mean Value theorem statement

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

where $f, g \in C^1$ and differentiable
in (a, b) and $c \in (a, b)$
 $g'(c) \neq 0$.

$$F(x) = f(x) + \lambda g(x)$$

$$F(a) = F(b)$$

$$\lambda = - \left[\frac{f(b) - f(a)}{g(b) - g(a)} \right]$$

$$F(x) = f(x) - \left[\frac{f(b) - f(a)}{g(b) - g(a)} \right] g(x)$$

F satisfies conditions of Rolle's theorem

i) $F(a) = F(b)$

ii) F is continuous in $[a, b]$
(because f, g are so)

iii) F is differentiable in (a, b)
(because f, g are so)

$$\therefore \exists c \in (a, b) \quad F'(c) = 0$$

$$\Rightarrow 0 = f'(c) - \lambda g'(c)$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$



Find the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$.

(15)

Candidates
must not write
on this margin

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\phi_1 = \frac{\partial u}{\partial x} = \frac{2 \cos 2x (\cosh 2y - \cos 2x) - \sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2 = \frac{\partial u}{\partial y} = \frac{-\sin 2x}{(\cosh 2y - \cos 2x)^2} [2 \sinh 2y]$$

$$f'(z) = \phi_1(z, 0) - i \phi_2(z, 0)$$

(Milne Thomson)

$$f'(z) = \frac{2 \cos 2z (1 - \cos 2z) - 2 \sin^2 2z}{(1 - \cos 2z)^2} - i \cdot 0$$

$$f'(z) = \frac{2 \cos 2z - 2}{(1 - \cos 2z)^2} = \frac{-2}{1 - \cos 2z} = \frac{-1}{\sin^2 z}$$

$$\frac{df}{dz} = -\operatorname{cosec}^2 z$$

$$f = +\cot z + C_1$$

SECTION-B

5. (a)

Form a partial differential equation by eliminating the arbitrary functions $5S$ and $5T$ from $z = yf(x) + xg(y)$. (10)

$$z = yf(x) + xg(y)$$

$$p = yf'(x) + g(y) + \cancel{xg'(y)}$$

$$q = f(x) + xg'(y)$$

$$r = \cancel{yf''(x)} + f'(x) + g'(y)$$

~~1000~~

$$xp + yq = xy(f'(x) + g'(y)) + xgy + yf(x)$$

$$\boxed{xp + yq = xy r + z}$$

5. (b)

Using Lagrange's interpolation formula, find a polynomial which passes the points $(0, -12)$, $(1, 0)$, $(3, 6)$, $(4, 12)$. (10)

x	0	1	3	4
$f(x)$	-12	0	6	12

$$\begin{aligned}
 f(x) = & \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-12) \\
 & + \frac{(x-0)(x-3)(x-4)}{(1)(-2)(-3)}(0) + \frac{(x-0)(x-1)(x-4)}{3(2)(-1)}(6) \\
 & + \frac{(x-0)(x-1)(x-3)}{4(3)(1)}(12)
 \end{aligned}$$

$$f(x) = x^3 - 7x^2 + 18x - 12$$

~~$$\begin{aligned}
 6x &= 16x^2 + 11x - 12 \\
 16x &= 16x^2 + 11x - 12
 \end{aligned}$$~~

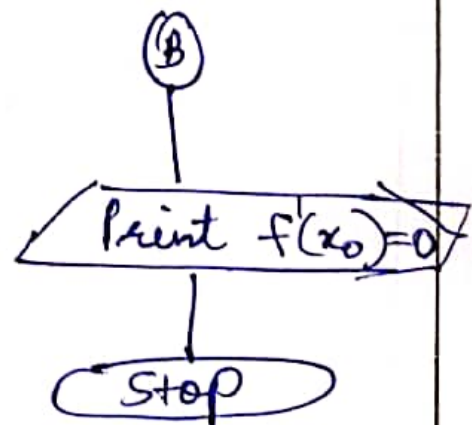
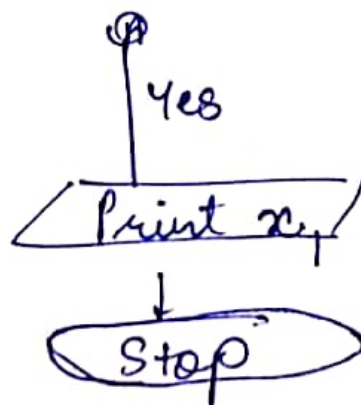
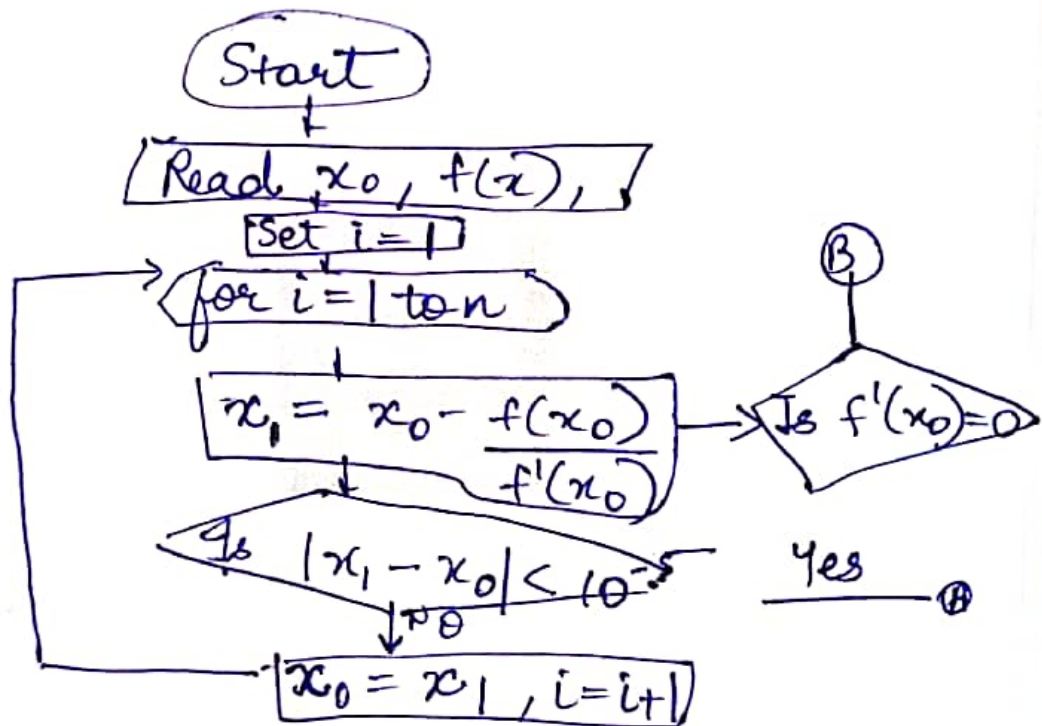
Draw a flow chart for solving equation $f(x) = 0$, correct to five decimal places by Newton-Raphson method. (10)

Candidate must write on this

$$f(x) = \cancel{x} =$$

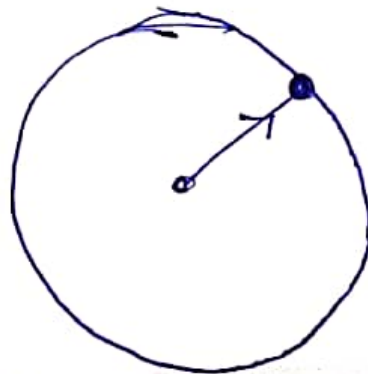
$$f(x+h) = f(x) + hf'(x)$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$



5. (d)

A heavy circular disk is revolving in a horizontal plane about the centre which is fixed. An insect of mass $\frac{1}{n}$ th that of the disk walks from the centre along a radius and flies away. Show that the final angular velocity is $\frac{n}{n+2}$ times the original angular velocity of the disk. (10)



Insect at periphery is initial moment when insect flies it is the final moment.

Conservation of ~~momentum~~ and angular momentum because no external torque applied

$$I\omega_i + \left(\frac{m}{n}\right)(r\omega_i)(r) = I\omega_f$$

↓
Disk's initial angular momentum

↓
 $m v r$
↓
angular momentum of insect

$$I = \frac{mr^2}{2}$$

$$\frac{\omega_f}{2} = \omega_i \left[\frac{1}{2} + \frac{1}{n} \right]$$

$$\omega_f = \omega_i \left[\frac{n+2}{n} \right]$$

5. (c) Does a fluid with velocity $\vec{q} = \left[z - \frac{2x}{r}, 2y - 3z - \frac{2y}{r}, x - 3y - \frac{2z}{r} \right]$ possess vorticity, where $\vec{q}(u,v,w)$ is the velocity in the Cartesian frame $\vec{r}(x,y,z)$ and $r^2 = x^2 + y^2 + z^2$? What is the circulation in the circle $x^2 + y^2 = 9, z=0$? (10)

$$\nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - \frac{2x}{r} & 2y - 3z - \frac{2y}{r} & x - 3y - \frac{2z}{r} \end{vmatrix}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$= \hat{i} \left(-3 + \frac{2z}{r^2} \frac{y}{r} + 3 - \frac{2y}{r^2} \frac{y}{r} \right)$$

$$+ \hat{j} \left(1 + \frac{2z}{r^2} \frac{x}{r} - 1 - \frac{2x}{r^2} \frac{z}{r} \right)$$

$$+ \hat{k} \left(\frac{2y}{r^2} \frac{x}{r} - \frac{2x}{r^2} \frac{y}{r} \right)$$

$$= 0$$

Hence no vorticity.

Circulation = ~~zero~~

$$= \frac{1}{2\pi} \int \hat{k} \cdot d\vec{S}$$

$\hat{k}$ is vorticity

$$= 0$$

Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0, y = 0$. (20)

$$\frac{dx}{dt} = f_p; \quad \frac{dy}{dt} = f_q; \quad \frac{dz}{dt} = p f_p + q f_q$$

$$\frac{dp}{dt} = -f_x - p f_z; \quad \frac{dq}{dt} = -f_y - q f_z$$

$$f: \quad z - p^2 + q^2$$

$$\frac{dx}{dt} = -2p \quad -i$$

$$\frac{dy}{dt} = 2q \quad -ii$$

$$\frac{dz}{dt} = -2p^2 + 2q^2 \quad -iii$$

$$\frac{dp}{dt} = -p \quad -iv$$

$$\frac{dq}{dt} = -q \quad -v$$

let $x = \lambda = f_1(\lambda), y = 0 = f_2(\lambda);$ $z = \frac{-\lambda^2}{4} = f_3(\lambda)$
 is initial

$$f'_3(\lambda) = p_0^* f'_1(\lambda) + q_0 f'_2(\lambda)$$

$$-\frac{\lambda}{2} = p_0$$

Also

$$f_3(\lambda) = p_0^2 - q_0^2 = \frac{-\lambda^2}{4}$$

$q_0^2 = \lambda^2 \therefore q_0 = \lambda$

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i, iv

$$q_0 = \frac{2}{\sqrt{2}}; p_0 = -\frac{\lambda}{2}$$

$$\frac{dx}{dt} = \frac{2dp}{dt} \Rightarrow x = 2p + c_1$$

$$\lambda = -2 \frac{\lambda}{2} + c_1$$

$$c_1 = 2\lambda$$

$$\Rightarrow x = 2p + 2\lambda \quad - \text{vi}$$

ii, v

$$\frac{dy}{dt} = -2 \frac{dq}{dt} \Rightarrow y = -2q + c_2$$

$$0 = -\frac{2\lambda}{\sqrt{2}} + c_2$$

$$c_2 = \sqrt{2}\lambda$$

$$y = -2q + \sqrt{2}\lambda \quad - \text{vii}$$

iv $\rightarrow$

$$\frac{dp}{p} = -dt \Rightarrow \log p = -c_3 t$$

$$p = c_4 e^{-t}$$

$$-\frac{\lambda}{2} = c_4 e^{-t} = c_4$$

$$p = -\frac{\lambda}{2} e^{-t} \quad - \text{viii}$$

v $\rightarrow$

$$\frac{dq}{q} = -dt \Rightarrow q = c_5 e^{-t}$$

$$\frac{\lambda}{\sqrt{2}} = c_5$$

$$q = \frac{\lambda}{\sqrt{2}} e^{-t} \quad - \text{ix}$$

vi, viii

$$x = 2\lambda \left(1 - \frac{e^{-t}}{2}\right) \quad - \text{(x)}$$

$$y = \sqrt{2}(\lambda)(1 - e^{-t}) \quad - \text{(xi)}$$

now

$$\frac{dz}{dt} = -2p^2 + 2q^2$$

$$= -\frac{\lambda^2}{2} e^{2t} + \lambda^2 e^{2t}$$

$$\frac{dz}{dt} = \frac{\lambda^2 e^{2t}}{2}$$

$$z = \frac{\lambda^2}{2} \left[\frac{e^{2t}}{2} \right] + C_6$$

$$-\frac{\lambda^2}{4} = \frac{\lambda^2}{4} + C_6 \Rightarrow C_6 = -\frac{\lambda^2}{2}$$

$$z = \frac{\lambda^2}{2} \left[\frac{e^{2t}}{2} - 1 \right] \quad \text{--- xii}$$

$$\frac{x}{y} = \sqrt{2} \frac{(1 - \frac{e^t}{2})}{(1 - e^t)}$$

$$(1 - e^t) x \sqrt{2} = \sqrt{2} (2 - e^t) y$$

$$e^t = \frac{2y - \sqrt{2}x}{y - x\sqrt{2}} \quad \text{--- xiii}$$

$$x = 2\lambda \left(1 - \frac{e^t}{2} \right)$$

$$= 2\lambda \left(1 - \frac{y - \frac{x}{\sqrt{2}}}{y - x\sqrt{2}} \right)$$

$$x = 2\lambda \left(\frac{\frac{x}{\sqrt{2}} - x\sqrt{2}}{y - x\sqrt{2}} \right)$$

$$\lambda = \frac{x(y - x\sqrt{2})}{(\sqrt{2}x - 2\sqrt{2}x)} = \frac{x\sqrt{2} - y}{\sqrt{2}}$$

$$\lambda = \frac{x - \frac{\sqrt{2}y}{2}}{-x'iv}$$

xii, xiii, xiv

$$3 = \frac{1}{4} \left(x - \frac{\sqrt{2}y}{2} \right)^2 \left[\left(\frac{2y - \sqrt{2}x}{y - x\sqrt{2}} \right)^2 - 2 \right]$$

$$3 = \frac{(\sqrt{2}x - y)^2}{84} \left[\frac{(2y - x)^2}{(y - x\sqrt{2})^2} - 1 \right]$$

$$= \frac{1}{4} \left[(\sqrt{2}y - x)^2 - (\sqrt{2}x - y)^2 \right]$$

$$= \frac{1}{4} [y^2 - x^2]$$

$$\boxed{43 = y^2 - x^2}$$

~~$$\left(\frac{-2x}{4} \right)^2 - \left(\frac{2y}{4} \right)^2 = \frac{(x-y)^2}{2} - \frac{y^2}{4}$$~~

6. (b)

A function f , defined on $[0, 1]$, is such that $f(0) = 0$, $f\left(\frac{1}{2}\right) = -1$, $f(1) = 0$. Find the quadratic polynomial $p(x)$ which agrees with $f(x)$ for $x = 0, \frac{1}{2}, 1$. If

$$\left| \frac{d^3 f}{dx^3} \right| \leq 1 \text{ for } 0 \leq x \leq 1, \text{ show that } |f(x) - p(x)| \leq \frac{1}{12} \text{ for } 0 \leq x \leq 1. \quad (15)$$

equidistant points, using Lagrange interpolation formulae

$$\begin{aligned} f(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) \\ &+ \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \\ &+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \\ &= \frac{(x - \frac{1}{2})(x - 1)}{\frac{1}{2}} \cdot 0 + \frac{(x)(x - 1)}{\frac{1}{2}(-\frac{1}{2})} (-1) \\ &\quad + 0 \end{aligned}$$

$$f(x) = 4x(x - 1)$$

$$p(x) = 4x^2 - 4x$$

error =

$$\frac{f^{(n)}(\xi)}{n!} (x-x_0)(x-x_1)\dots(x-x_{n-1})$$

$$n=3$$

$$\text{error} = \frac{f^{(3)}(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2)$$

$$\leq \left| \frac{f^{(3)}(\xi)}{3!} \right| |(x-x_0)(x-x_1)(x-x_2)|$$

$$\leq \frac{|f^{(3)}(\xi)|}{6} |x-x_0| |x-x_1| |x-x_2|$$

$$\leq \frac{1}{6} [1] \left[x - \frac{1}{2} \right] [1]$$

$$\leq \frac{1}{6} \left[1 \right] \left[\frac{1}{2} \right] [1]$$

$$\text{error} \leq \frac{1}{12}$$

$$\begin{aligned} \max |x-0| &= 1 \\ \max |x-\frac{1}{2}| &= \frac{1}{2} \text{ at } x=0,1 \\ \max |x-1| &= 1 \text{ at } x=0 \end{aligned}$$

$$\text{error} = |f(x) - p(x)| \leq \frac{1}{12}$$

Consider the integers 13, 8 and their 8 bit binary representations. Perform the following bitwise operations and express the results in decimal system:

- (i) $13 \& 20$ (bitwise AND)
- (ii) $13 | 20$ (bitwise OR)
- (iii) $13 \wedge 20$ (bitwise XOR)
- (iv) ~ 20 (bitwise Complement)

(15)

$$13 = 00001101$$

$$20 = 00010100$$

i) $13 \text{ AND } 20 = (13) \cdot (20)$

$$\begin{array}{r} 00001101 \\ 00010100 \\ \hline 00000100 \\ \hline = (4)_{10} \end{array}$$

ii) $13 \text{ OR } 20$

$$\begin{array}{r} 00001101 \\ 00010100 \\ \hline 00011101 \\ \hline = (29)_{10} \end{array}$$

iii) $13 \wedge 20$

$$\begin{array}{r} 00001101 \\ 00010100 \\ \hline 11111011 \\ \hline = (250)_{10} \\ = (235)_{10} \end{array}$$

8. (a)

Consider a uniform flow U_0 in the positive x -direction. A cylinder of radius a is located at the origin. Find the stream function and the velocity potential. Find also the stagnation points. (15)

$$\longrightarrow U_0$$

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}$$

$\Rightarrow \psi = -U_0 y$ is stream function for uniform flow

Also $\phi = -U_0 x$

$$\begin{aligned} w &= \phi + i\psi = -U_0(x + iy) \\ &= -U_0 z \end{aligned}$$

Cylinder is inserted $\rightarrow w'$ is new complex potential.

$$\begin{aligned} w' &= w + i\bar{w}\left(\frac{a^2}{z^2}\right) \\ &= -U_0 z + \left(-U_0 \frac{a^2}{z^2}\right) \\ w' &= -U_0 \left(z + \frac{a^2}{z}\right) \end{aligned}$$

Now let $z = re^{i\theta}$

$$\begin{aligned} w' &= -U_0 \left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right) \\ &= -U_0 \left[\cos\theta \left(r + \frac{a^2}{r}\right) + i\sin\theta \left(r - \frac{a^2}{r}\right)\right] \end{aligned}$$

$$\omega' = \phi + i\psi$$

$$\phi = -U_0 \cos\theta \left(1 + \frac{a^2}{r^2}\right)$$

$$\psi = -U_0 \left(r - \frac{a^2}{r}\right) \sin\theta$$

For stagnation points

$$\left| \frac{d\omega'}{dz} \right| = 0 = -u + i v$$

$$= \left(-U_0 + \frac{a^2 U_0}{z^2} \right) = 0$$

$$z = r e^{i\theta}$$

$$\left(1 - \frac{a^2}{r^2} e^{-2i\theta} \right) = 0$$

$$\left(1 - \frac{a^2}{r^2} \cos 2\theta \right)^2 + \left(\frac{a^2}{r^2} \sin 2\theta \right)^2 = 0$$

$$1 - \frac{2a^2}{r^2} \cos\theta + \frac{a^4}{r^2} = 0$$

$$1 + \frac{a^4}{r^4} = \frac{2a^2}{r^2} \cos\theta$$

$$\left(1 - \frac{a^2}{r^2} \cos 2\theta \right) = 0 ; \quad \frac{a^2}{r^2} \sin 2\theta = 0$$

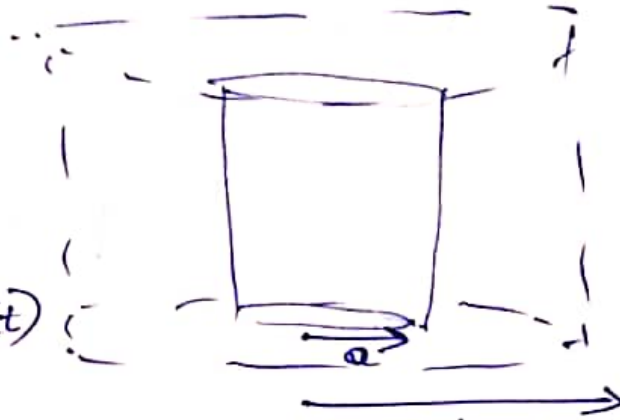
Stagnation points:

$$(r, \theta) = (a, 0); (a, \pi) \Rightarrow \theta = 0, \pi/2, \pi, 3\pi/2$$

$$r = a, \infty, a, \infty$$

The space between two infinitely long coaxial cylinders of radii a and b ($b > a$) respectively is filled by homogeneous fluid, of density ρ . The inner cylinder is suddenly moved with velocity v perpendicular to this axis, the outer being kept fixed. Show that the resulting impulsive pressure on a length l of inner cylinder

is, $\pi \rho a^2 l \frac{b^2 + a^2}{b^2 - a^2} v$. (20)



$$xv = \text{constant} = F(t)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = F - \frac{1}{\rho} \frac{dp}{dx}$$

$$\frac{1}{x} F'(t) + \frac{1}{2} \frac{\partial v^2}{\partial x} = F - \frac{1}{\rho} \frac{dp}{dx}$$

$$F'(t) \log x + \frac{v^2}{2} = -\frac{p}{\rho} + C_1$$

t	x	v	p
0	a	v	p
0	b	0	π

$$F(t) = av$$

$$\frac{\partial \bar{w}}{\partial r} = -\rho v(r=a)$$

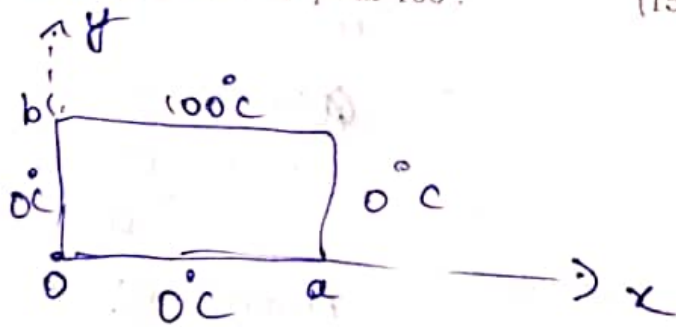
$$v_r = av$$

$$v_r = \frac{av}{r}$$

$$\bar{w} = -\rho av \log r + C_1$$

(c)

Find the steady state temperature distribution in a thin rectangular plate bounded by the lines $x=0, x=a, y=0$ and $y=b$. The edges $x=0, x=a$ and $y=0$ are kept at temperature zero while the edge $y=b$ is kept at 100° . (15)



Steady state temperature distribution

is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = k^2$$

Variable separation

$$u(x, y) = X(x) Y(y)$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k^2$$

$$k=0 \Rightarrow X(x)=0 \Rightarrow u=0$$

absurd.

$$k^2 = \lambda^2$$

$$X''(x) = \lambda^2 X(x)$$

$$X(x) = A e^{\lambda x} + B e^{-\lambda x}$$

$$X(0) = X(a) = 0 \Rightarrow X(x) = 0$$

$$\Rightarrow u(x) = 0 \text{ absurd}$$

$$k^2 = -\lambda^2$$

$$X(x) = A \sin \lambda x + B \cos \lambda x$$

$$B=0$$

$$X(a) = 0$$

$$\Rightarrow A = 0 \quad \text{absurd}$$

$$\Rightarrow \lambda = n\pi$$

$$\lambda = \frac{n\pi}{a}$$

$$X(x) = A \sin \frac{n\pi x}{a}$$

$$Y(y) = C e^{\lambda y} + D e^{-\lambda y}$$

$$Y(0) = 0 \Rightarrow C + D = 0$$

$$Y(a) = 100 \Rightarrow C e^{\lambda a} + D e^{-\lambda a} = 100$$

$$D(-e^{\lambda a} + e^{-\lambda a}) = 100$$

$$D(e^{-n\pi} - e^{n\pi}) = 100$$

$$D = \frac{100 e^{n\pi}}{(1 - e^{2n\pi})} = -C$$

$$Y(y) = \frac{100 e^{n\pi}}{1 - e^{2n\pi}} \left(-e^{+\frac{n\pi}{a} y} + e^{-\frac{n\pi}{a} y} \right)$$

$$u(x, y) = X(x) Y(y)$$

$$Y(y) = C \left(e^{\frac{n\pi}{a} y} - e^{-\frac{n\pi}{a} y} \right)$$

$$u(x, y) = AC \sin \frac{n\pi x}{a} \left(e^{\frac{n\pi}{a} y} - e^{-\frac{n\pi}{a} y} \right)$$

$$= E \sin \frac{n\pi x}{a} \left(e^{\frac{n\pi}{a} y} - e^{-\frac{n\pi}{a} y} \right)$$

$$E = \frac{2}{a} \int_0^a 100 \sin \frac{n\pi x}{a} dx$$

$$= 0 \quad n = 2m$$

$$\frac{400}{a} \quad n = 2m + 1$$

$$E = \frac{400}{a (e^{n\pi \frac{b}{a}} - e^{-n\pi \frac{b}{a}})}$$

$$u(n, y) = \sum_{n=1}^{\infty} \left[\frac{400}{a (e^{n\pi \frac{b}{a}} - e^{-n\pi \frac{b}{a}})} \right] \left[\frac{\sin \frac{n\pi x}{a}}{a} \right] \left[(e^{n\pi \frac{y}{a}} - e^{-n\pi \frac{y}{a}}) \right]$$