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NEXT IAS

(To be filled by candidate)

Name : ANNU KOUSIK NARASIMHA T-10 | 81

Rol No. : OAC24MTTA003 Date of Exam.: 20-08-2024

Mobile No. ..... Email: -com

Exam Centre : Old Rajinder Nagar [✓] Bhopal [ ] Jaipur [ ]
Mukherjee Nagar [ ] Prayagraj [ ] Online [ ]

CSE (MAINS) TEST SERIES-2024

Test-10	MATHEMATICS OPTIONAL
Dated : 18-08-2024	

Time Allowed : Three Hours | Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

*There are **EIGHT** questions divided in **Two sections** and printed in **ENGLISH**.*

*Candidate has to attempt **FIVE** questions in all.*

***Question No. 1 and 5** are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.*

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	15			
2. (c)	20	20			
3. (a)	24	20			
3. (b)	28	15			
3. (c)	31	15			
4. (a)	34	20			
4. (b)	38	15			
4. (c)	41	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	15			
7. (c)	70	20			
8. (a)	74	15			
8. (b)	77	20			
8. (c)	81	15			
Grand Total					

Remarks

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Observations:

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SECTION-A

1. (a) Show that $(1+i)$ is prime element in the ring R of Gaussian integers. (10)

To show : $(1+i)$ is prime element.

$1+i$ is not a unit $(\because (1+i)^{-1} = \frac{1}{2} - \frac{i}{2} \notin R)$

Let $(1+i) \mid (a+ib)(c+id)$ where $a, b, c, d \in \mathbb{Z}$

$\Rightarrow (1-i) \mid (a-ib)(c-id)$ (Taking conjugates)

Multiplying,

$(1+i)(1-i) \mid (a+ib)(a-ib)(c+id)(c-id)$

$\Rightarrow 2 \mid (a^2+b^2)(c^2+d^2) \Rightarrow 2 \mid a^2+b^2$ or $2 \mid c^2+d^2$

$(\because 2$ is prime$)$

Let $2 \mid (a^2+b^2)$ ——— (1)

Now $2 = (1+i)(1-i)$

$\Rightarrow (1+i) \mid 2$ ——— (2)

From (1) & (2), $(1+i) \mid a^2+b^2$

$\Rightarrow (1+i) \mid a+ib$ $(\because 1+i \mid (a+ib)(c+id)$ & $1+i \mid (a+ib)(a-ib))$

(or) $\Rightarrow (1+i)$ is prime element of R.

(Or) $(1+i) \mid c+id$

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1. (b) If H is a subgroup of a group G such that $x^2 \in H$ for every $x \in G$, then prove that H is a normal subgroup of G. (10)

Let $x \in G \Rightarrow x^2 \in H$

Then, consider xhx^{-1} where $h \in H$

To show $khk^{-1} \in H \quad \forall k \in G, h \in H$

Let $h^2 \in H \Rightarrow h \cdot h^2 \in H$

Consider

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1. (c) Find the extreme value of $f(x, y, z) = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$. (10)

Let $\phi(x,y,z) = x^2+y^2-5$; $\psi(x,y,z) = x+z-1$

Finding extreme values by Lagrange multiplier method;

$$d(f+\lambda\phi+\mu\psi) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} + \mu\frac{\partial \psi}{\partial x} = 0$$

$$\Rightarrow 2 + 2x\lambda + \mu = 0 \quad \text{---} (1)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} + \mu\frac{\partial \psi}{\partial y} = 0$$

$$\Rightarrow 3 + 2y\lambda + 0 = 0 \quad \text{---} (2)$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} + \mu\frac{\partial \psi}{\partial z} = 0$$

$$\Rightarrow 1 + \lambda(0) + \mu(1) = 0$$

$$\Rightarrow \mu = -1 \quad \text{---} (3).$$

$\mu = -1 \Rightarrow$ from (1); $2x\lambda = -1$

$$\Rightarrow \boxed{x = \frac{-1}{2\lambda}}$$

From (2), $\boxed{y = \frac{-3}{2\lambda}}$. Now $x^2+y^2=5$

$$\Rightarrow \frac{1}{4\lambda^2} + \frac{9}{4\lambda^2} = 5 \Rightarrow 4\lambda^2 = 2$$

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$\Rightarrow \lambda = \pm \frac{1}{\sqrt{2}}$

$\lambda = \frac{1}{\sqrt{2}}$: $x = \frac{-1}{2\lambda} = -\frac{1}{2}\sqrt{2} = -\frac{1}{\sqrt{2}}$

$y = \frac{-3}{2\lambda} = \frac{-3}{\sqrt{2}}$

$z = 1 - x = 1 + \frac{1}{\sqrt{2}}$

$2x + 3y + z = f = 2\left(\frac{-1}{\sqrt{2}}\right) + 3\left(\frac{-3}{\sqrt{2}}\right) + \left(1 + \frac{1}{\sqrt{2}}\right)$

$= \frac{-2}{\sqrt{2}} - \frac{9}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 = \frac{-10}{\sqrt{2}} + 1 = 1 - 5\sqrt{2}$

$\therefore f = 1 - 5\sqrt{2}$ is one extremum

$\lambda = \frac{-1}{\sqrt{2}} \Rightarrow x = \frac{-1}{2\lambda} = \frac{1}{\sqrt{2}}$; $y = \frac{-3}{2\lambda} = \frac{3}{\sqrt{2}}$

$z = 1 - x = 1 - \frac{1}{\sqrt{2}}$.

$2x + 3y + z = f = 2\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{3}{\sqrt{2}}\right) + \left(1 - \frac{1}{\sqrt{2}}\right)$

$= 1 + 5\sqrt{2}$

$\therefore f = 1 + 5\sqrt{2}$ is other extremum.

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1. (d) Find the residue of $\frac{z^3}{z^2-1}$ at the infinity. (10)

Let $f(z) = \frac{z^3}{z^2-1}$.

Residue of $f(z)$ at infinity = Residue of $f(1/z)$ at $z=0$

$\Rightarrow f(1/z) = \frac{1/z^3}{\frac{1}{z^2}-1} = \frac{1 \times z^2}{z^3(1-z^2)} = \frac{1}{z(1-z^2)}$

Residue at $z=0$ is

$R = \lim_{z \to 0} (z-0) \times \frac{1}{z(1-z^2)} = 1$

(∵ Residue at a pole of order n is $\lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z) \times \frac{1}{(n-1)!}$)

∴ Residue of $f(z)$ at infinity $= 1$

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1. (e) Show that the function $e^{-\frac{1}{z^2}}$ has no singularities. (10)

Given fn $f(z) = e^{-1/z^2}$

To show $f(z)$ defined $\forall z$

$$e^{-1/z^2} = \frac{1}{e^{1/z^2}} = \frac{1}{1 + \frac{1}{z^2} + \frac{1}{2!}\left(\frac{1}{z^2}\right)^2 + \cdots + \frac{1}{n!}\left(\frac{1}{z^2}\right)^n + \cdots}$$

The fraction $\frac{1}{1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{1}{z^2}\right)^n}$

is defined $\forall z$

and $1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{1}{z^2}\right)^n \neq 0 \ \forall z$

Hence e^{-1/z^2} has no singularities.

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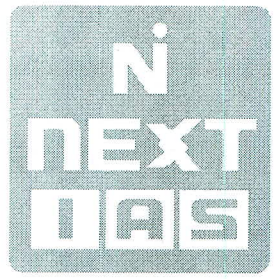
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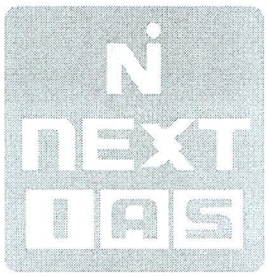
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2. (a) Let $o(G) = 108$. Show that there exists a normal subgroup of order 27 or 9. (15)



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2. (b) Show that the sequence $\{\tan^{-1} nx\}, x \geq 0$ is uniformly convergent on any interval $[a,b], a > 0$ but is only pointwise convergent on [0, b]. (15)



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2. (c) Prove that $\int_0^\infty \frac{\sin mx}{x\left(x^2+a^2\right)^2}dx = \frac{\pi}{2a^4} - \frac{\pi e^{-ma}}{4a^3}\left(m+\frac{2}{a}\right), m>0, a>0.$ (20)

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3. (a) Prove that x^2+x+4 is irreducible over F, the field of integer modulo 11 and prove further that $\frac{F[X]}{(x^2+x+4)}$ is a field having 121 elements. (20)

To show : (x^2+x+4) is irreducible over $(Z_{11}, +_{11}, \times_{11})$

Calculating (x^2+x+4) values $\forall$ values of $(Z_{11}, +_{11}, \times_{11})$

$x=0 \Rightarrow x^2+x+4 = 0 \times_{11} 0 +_{11} 0 +_{11} 4 = 4$

$x=1 \Rightarrow x^2+x+4 = 1 \times_{11} 1 +_{11} 1 +_{11} 4 = 6$

$x=2 \Rightarrow x^2+x+4 = (10)_{11} = 10$

$x=3 \Rightarrow x^2+x+4 = (16)_{11} = 5$

$x=4 \Rightarrow x^2+x+4 = (24)_{11} = 2$

$x=5 \Rightarrow x^2+x+4 = (34)_{11} = 1$

$x=6 \Rightarrow x^2+x+4 = (46)_{11} = 2$

$x=7 \Rightarrow x^2+x+4 = (60)_{11} = 5$

$x=8 \Rightarrow x^2+x+4 = (76)_{11} = 10$

$x=9 \Rightarrow x^2+x+4 = (94)_{11} = 6$

$x=10 \Rightarrow x^2+x+4 = (114)_{11} = 4$

$\therefore x^2+x+4 \neq 0 \quad \forall x \in Z_{11}$

Hence x^2+x+4 is irreducible over Z_{11}

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x^2+x+4 is irreducible over $F[x]$

$\Rightarrow \langle x^2+x+4\rangle$ is maximal ideal of $F[x]$

$\Rightarrow \frac{F[x]}{\langle x^2+x+4\rangle}$ is a field. Let $I = \langle x^2+x+4\rangle$

~~By division~~

Elements of $\frac{F[x]}{\langle x^2+x+4\rangle}$ are of the form $\{k(x)+I$ where $k \in F\}$

By division algorithm,

$k'(x) = p(x)I + r(x)$ where $k'(x) \in \frac{F}{\langle I\rangle}$

and $r(x) = 0$ or $\deg(r(x)) < \deg(p(x))$

$\Rightarrow k(x) = p(x)I + r(x)$

$k(x) - r(x) \in I \Rightarrow \boxed{k(x)+I = r(x)+I}$

$\deg(r(x)) < \deg(p(x)) \Rightarrow \deg(r(x)) < 2$

$\Rightarrow \deg(r(x)) = 1$ or zero.

Let $r(x) = ax+b$.

Now the possible elements of r

$\Rightarrow r = ax+b$ where $a \in [0,10], b \in [0,10]$

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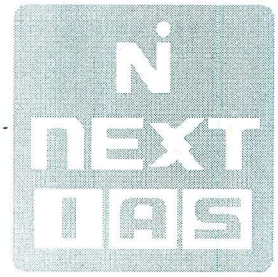
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No of elements of field

= No of possible values of a × No of possible values of b

= 11 × 11 = 121

$\therefore \dfrac{F(x)}{\langle x^2+x+4\rangle}$ consists of 121 elements.



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3. (b) Prove Cauchy's Mean value theorem. (15)

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Consider a function $h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g(x)$

Such that f, g are continuous over $[a,b]$ & differentiable over (a,b)

Also Let $g(b) \neq g(a)$.

Then $h(x)$ obeys rolle's theorem.

$\Rightarrow \exists\ c \in (a,b)$ such that $h'(c) = 0$.

$\Rightarrow h = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g(x)$

$\Rightarrow h'(x) = f'(x) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g'(x)$

$h'(c) = 0$

$\Rightarrow f'(c) = g'(c) \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)$

$\Rightarrow \boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$ for some $c \in (a,b)$

Hence, proved.

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3. (c) Find the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$. (15)

Let $w = \phi + i\psi$. Then ϕ, ψ obeys cauchy riemann eqns

$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$; $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$.

Given that $\phi = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}.$$

$$\frac{\partial \phi}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - (\sin 2x)(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial \phi}{\partial x} = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial \phi}{\partial y} = \frac{-\sin 2x (2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + -i\frac{\partial \phi}{\partial y}$$

$$\Rightarrow \frac{dw}{dz} = \frac{2(\cos 2x \cosh 2y - 1)}{(\cosh 2y - \cos 2x)^2} + \frac{i\, 2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

Substituting $y = 0$, $x = z$, we get

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$$\frac{dw}{dz} = \frac{2(\cos 2z - 1)}{(\cos 2z - 1)^2} = \frac{2}{\cos 2z - 1}$$

$$\Rightarrow \frac{dw}{dz} = \frac{2}{1 - 2\sin^2 z - 1} = -\operatorname{cosec}^2 z$$

$$\Rightarrow dw = -\operatorname{cosec}^2 z \, dz$$

Integrating on both sides,

$$w = \int -\operatorname{cosec}^2 z \, dz$$

$$\Rightarrow \boxed{w = \cot z + c}$$ is the reqd analytic fn.

where $z = x + iy$

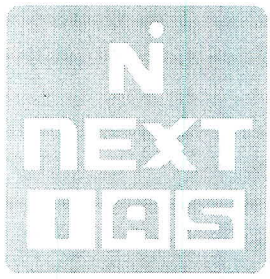
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4. (a) If (x, y, z) be the lengths of perpendiculars drawn from any interior point P of a triangle ABC on the sides BC, CA and AB respectively, then find the minimum value of $x^2 + y^2 + z^2$, the sides of the triangle ABC being a, b, c. (20)



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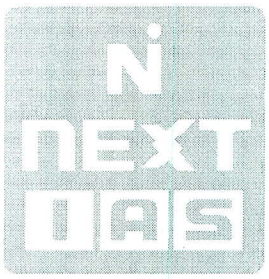
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4. (b) An automobile dealer wishes to put four repairmen R_1, R_2, R_3 and R_4 to four different jobs J_1, J_2, J_3 and J_4. But R_3 cannot do the job J_2. The dealer has estimated the numbers of man hours that would be required for each job man on one-one basis as given in the following table:

	R_1	R_2	R_3	R_4
J_1	6	2	3	4
J_2	9	7	–	5
J_3	6	4	7	5
J_4	6	8	8	9

Formulate the above as Linear Programming Problem. (15)



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4. (c) Prove that a basic feasible solution to a linear programming problem must correspond to an extreme point of the set of all feasible solutions. (15)



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SECTION-B

5. (a) Form a partial differential equation by eliminating the arbitrary functions $5S$ and $5T$ from $z=yf(x)+xg(y)$. (10)

Let us assume the functions f, g are to be eliminated.

$$z = yf(x) + xg(y) \quad \text{—(1)}.$$

Diff (1) partially both sides w.r.t x,

$$\frac{\partial z}{\partial x} = p = yf'(x) + g(y) \quad \text{—(2)}$$

Diff (1) partially both sides w.r.t y,

$$\frac{\partial z}{\partial y} = q = f(x) + xg'(y) \quad \text{—(3)}.$$

Diff (2) partially w.r.t y,

$$pq = f'(x) + g'(y) \quad \text{—(4)}.$$

Multiplying (2) on both sides with x and (3) on both sides with y and adding,

$$px + qy = xyf'(x) + xg(y) + yf(x) + xyg'(y)$$

$$\Rightarrow px + qy = xy\left(f'(x) + g'(y)\right) + \left(xg(y) + yf(x)\right)$$

$$\Rightarrow px + qy = (xy)pq + z$$

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$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - xy\frac{\partial^2 z}{\partial x \partial y} = z$$

is the reqd PDE



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5. (b) Using Lagrange's interpolation formula, find a polynomial which passes the points (0, –12), (1, 0), (3, 6), (4, 12). (10)

Let the values be

$x_0 = 0, \quad x_1 = 1, \quad x_2 = 3, \quad x_3 = 4$

$y_0 = -12, \quad y_1 = 0, \quad y_2 = 6, \quad y_3 = 12$

The Lagranges polynomial is given by :

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

substituting the values,

$$y(x) = \frac{(x-1)(x-3)(x-4)}{-1 \times -3 \times -4} \times (-12) + \frac{x(x-3)(x-4)}{1 \times -2 \times -3} \times 0$$

$$+ \frac{x(x-1)(x-4)}{3 \times 2 \times -1} \times 6 + \frac{x(x-1)(x-3)}{4 \times 3 \times 1} \times 12$$

$$\Rightarrow y(x) = (x-1)(x-3)(x-4) - x(x-1)(x-4) + x(x-1)(x-3)$$

$$\Rightarrow y(x) = (x^3 - 8x^2 + 19x - 12) - (x^3 - 5x^2 + 4x) + (x^3 - 4x^2 + 3x)$$

$$\Rightarrow \boxed{y(x) = x^3 - 7x^2 + 18x - 12}$$

is the reqd polynomial

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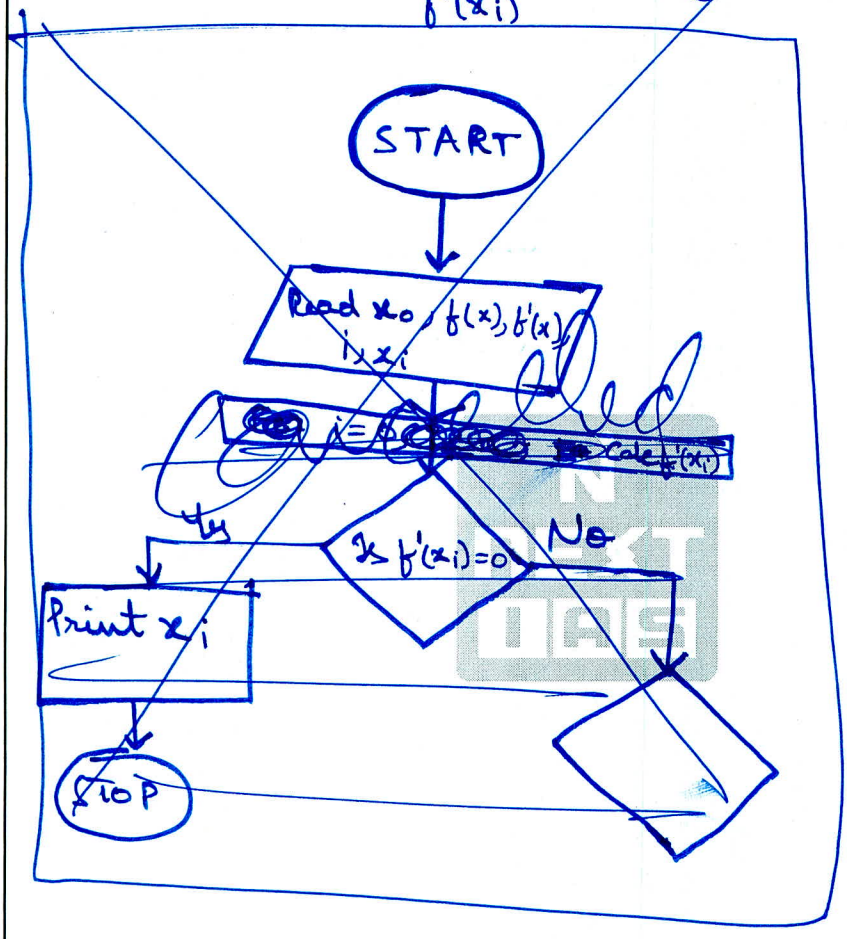
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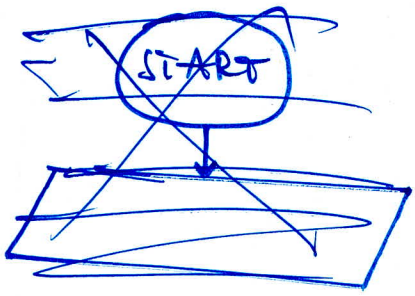
5. (c) Draw a flow chart for solving equation $F(x) = 0$, correct to five decimal places by Newton-Raphson method. (10)

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Newton raphson method

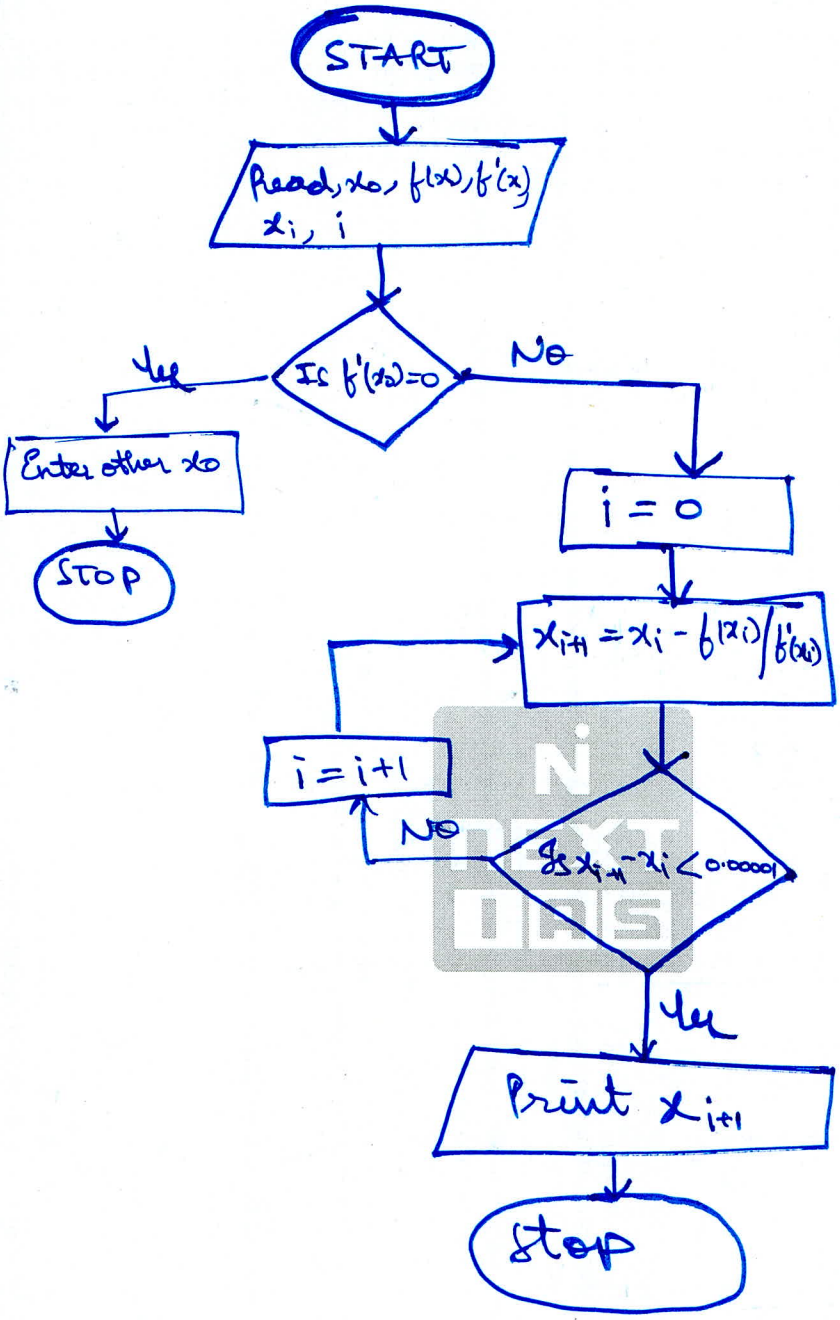
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (x_{i+1} - x_i < 10^{-5})$$





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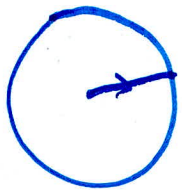
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5. (d) A heavy circular disk is revolving in a horizontal plane about the centre which is fixed. An insect of mass $\frac{1}{n}th$ that of the disk walks from the centre along a radius and flies away. Show that the final angular velocity is $\frac{n}{n+2}$ times the original angular velocity of the disk. (10)



By conservation of angular momentum,

$mk^2\dot{\theta} = \text{constant}$.

Initially (before flight): Let angular $\vec{v} = \omega$

After flight, Let angular $\vec{v} = \omega'$

$$\frac{m}{n}r^2\omega^2 + \frac{m}{2}r^2\omega^2 \geq \frac{m}{2}r^2\omega'$$

$$\Rightarrow \frac{m}{n}r^2\omega + \frac{mr^2\omega}{2} = \frac{mr^2\omega'}{2}$$

$$\Rightarrow \frac{\omega}{n} + \frac{\omega}{2} = \frac{\omega'}{2} \Rightarrow \frac{\omega'}{2} = \omega\left(\frac{n+2}{2n}\right)$$

$$\Rightarrow \boxed{\omega = \left(\frac{n}{n+2}\right)\omega'}$$

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5. (e) Does a fluid with velocity $\vec{q}=\left[z-\frac{2x}{r}, 2y-3z-\frac{2y}{r}, x-3y-\frac{2z}{r}\right]$ possess vorticity, where $\vec{q}(u,v,w)$ is the velocity in the Cartesian frame $\vec{r}(x,y,z)$ and $r^2=x^2+y^2+z^2$? What is the circulation in the circle $x^2+y^2=9, z=0$? (10)

If the fluid possess vorticity then $\nabla \times q \neq 0$.

$$\nabla \times q = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z-\frac{2x}{r} & 2y-3z-\frac{2y}{r} & x-3y-\frac{2z}{r} \end{vmatrix}$$

$\frac{\partial r}{\partial x}=\frac{x}{r}, \frac{\partial r}{\partial y}=\frac{y}{r}$

$\frac{\partial r}{\partial z}=\frac{z}{r}$

$$= i\left(\frac{\partial}{\partial y}\left(x-3y-\frac{2z}{r}\right)-\frac{\partial}{\partial z}\left(2y-3z-\frac{2y}{r}\right)\right)+j\left(\frac{\partial}{\partial x}\left(x-3y-\frac{2z}{r}\right)-\frac{\partial}{\partial z}\left(z-\frac{2x}{r}\right)\right)$$

$$+k\left(\frac{\partial}{\partial x}\left(2y-3z-\frac{2y}{r}\right)-\frac{\partial}{\partial y}\left(z-\frac{2x}{r}\right)\right)$$

$$= i\left(-3+\frac{2zy}{r^3}+3-\frac{2yz}{r^3}\right)+j\left(1+\frac{2xz}{r^3}-1-\frac{2xz}{r^3}\right)$$

$$+k\left(\frac{2xy}{r^3}-\frac{2xy}{r^3}\right)=0i+0j+0k.$$

$\therefore \nabla \times q = 0$ $\therefore$ The fluid doesnot possess vorticity

Circulation $= \oint_C q \cdot dr$.

$\nabla \times q$ is zero $\Rightarrow q$ is Conservative

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$\Rightarrow \oint_C q \cdot dr = 0$

$\therefore$ circulation $= 0$ on the closed circle $z=0, x^2+y^2=9$.

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6. (a) Determine the characteristics of the equation $z=p^2-q^2$ and find the integral surface which passes though the parabola $4z+x^2=0, y=0$. (20)

Let $f(x,y,z,p,q) = p^2-q^2-z$.

Let $y_0=0$, $x_0=2\lambda$, $z_0=-\frac{1}{4}x_0^2=-\lambda^2$

Let p_0, q_0 be the initial values of p, q at $t=0$.

$\Rightarrow dz_0 = p_0\,dx_0 + q_0\,dy_0$

$\Rightarrow -2\lambda = 2p_0 \Rightarrow p_0=-\lambda$

Also $z_0 = p_0^2 - q_0^2 \Rightarrow -\lambda^2 = \lambda^2 - q_0^2$

$\Rightarrow q_0 = \sqrt{2}\lambda$

The Cauchys char eqns are given by

$$\frac{dx}{dt} = f_p \;;\; \frac{dy}{dt} = f_q \;;\; \frac{dz}{dt} = pf_p + qf_q;$$

$$\frac{dp}{dt} = -f_x - pf_z \;;\; \frac{dq}{dt} = -f_y - qf_z.$$

$f_x = f_y = 0$; $f_z = -1$, $f_p = 2p$; $f_q = -2q$.

$$\frac{dx}{dt} = f_p \Rightarrow \boxed{\frac{dx}{dt} = 2p} \quad \text{---} (1).$$

$$\frac{dy}{dt} = f_q \Rightarrow \boxed{\frac{dy}{dt} = -2q} \quad \text{---} (2).$$

$$\frac{dz}{dt} = pf_p + qf_q = 2p^2 - 2q^2 = 2z$$

$$\therefore \boxed{\frac{dz}{dt} = 2z} \quad \text{---} (3).$$

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$\frac{dp}{dt} = -(f_x + p f_z) = -(0+p) = p$

$\frac{dq}{dt} = -(f_y + q f_z) = -(0-q) = q$

$\therefore \frac{dp}{dt} = p$ — (4) & $\frac{dq}{dt} = q$ — (5).

Now $\frac{dz}{dt} = 2z \Rightarrow$ on integrating,

$z = ke^{2t}$; $z_0 = -\lambda^2 = k_1$

$\therefore z = -\lambda^2 e^{2t}$ — (6)

$\frac{dp}{dt} = p \Rightarrow p = k_2 e^t$; $p_0 = -\lambda = k_2$

$\therefore p = -\lambda e^t$ — (7)

$\frac{dq}{dt} = q \Rightarrow q = k_3 e^t$; $q_0 = -\sqrt{2}\lambda = k_3$

$\therefore q = \sqrt{2}\lambda e^t$ — (8)

$\frac{dx}{dt} = 2p = -2\lambda e^t$ (From (1))

$\Rightarrow x = -2\lambda e^t + k_4$

$x_0 = 2\lambda = -2\lambda + k_4 \Rightarrow k_4 = 4\lambda$

$\therefore x = 4\lambda - 2\lambda e^t$ — (9).

$\frac{dy}{dt} = -2q = -\sqrt{2}\lambda e^t$.

$\Rightarrow y = -\sqrt{2}\lambda e^t + k_5$

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$y_0 = 0 = -\sqrt{2}\lambda + k \Rightarrow k = \sqrt{2}\lambda$

$\Rightarrow \boxed{y = \sqrt{2}\lambda - \sqrt{2}\lambda e^t}$ — (10)

$\frac{(9)}{(10)} \Rightarrow \frac{x}{y} = \frac{2\lambda(2-e^t)}{\sqrt{2}\lambda(1-e^t)} = \frac{\sqrt{2}(2-e^t)}{(1-e^t)}$

$\Rightarrow x - xe^t = 2\sqrt{2}y - \sqrt{2}ye^t$

$\Rightarrow (x - 2\sqrt{2}y) = e^t(x - \sqrt{2}y)$

$\Rightarrow \boxed{e^t = \left(\frac{x-2\sqrt{2}y}{x-\sqrt{2}y}\right)}$

From 9, $x = 2\lambda(2-e^t)$

$\Rightarrow x = 2\lambda\left(2 - \frac{x-2\sqrt{2}y}{x-\sqrt{2}y}\right)$

$\Rightarrow x = 2\lambda\left(\frac{2x - 2\sqrt{2}y - x + 2\sqrt{2}y}{x-\sqrt{2}y}\right) = 2\lambda\left(\frac{x}{x-\sqrt{2}y}\right)$

$\Rightarrow x = \frac{2\lambda x}{x-\sqrt{2}y} \Rightarrow \boxed{\lambda = \frac{1}{2}(x-\sqrt{2}y)}$

Now, $z = -\lambda^2 e^{2t}$

$\Rightarrow z = -(\lambda e^t)^2 = -\left(\frac{1}{2}(x-\sqrt{2}y) \times \frac{(x-2\sqrt{2}y)}{(x-\sqrt{2}y)}\right)^2$

$\Rightarrow \boxed{z = \frac{-1}{4}(x-2\sqrt{2}y)^2}$

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$\therefore$ The eqn of surface is

$$4z + (x - 2\sqrt{2}y)^2 = 0$$



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6. (b) A function f, defined on [0, 1], is such that $f(0)=0,\ f\left(\frac{1}{2}\right)=-1, f(1)=0$. Find the quadratic polynomial $p(x)$ which agrees with $f(x)$ for $x=0,\frac{1}{2},1$. If $\left|\frac{d^3 f}{dx^3}\right| \le 1$ for $0 \le x \le 1$, show that $|f(x)-p(x)| \le \frac{1}{12}$ for $0 \le x \le 1$. (15)

Writing the quadratic polynomial by Lagrange's method,

$$p(x) = \frac{(x-1/2)(x-1)}{(0-1/2)(0-1)} f(0) + \frac{(x-0)(x-1)}{(\frac{1}{2}-0)(\frac{1}{2}-1)} f(1/2)$$

$$+ \frac{(x-0)(x-1/2)}{(1-0)(1-1/2)} f(1)$$

$$\Rightarrow p(x) = \frac{x(x-1)}{(-1/4)}(-1) = 4x(x-1)$$

$\therefore p(x) = 4x^2 - 4x$ is the reqd quadratic polynomial.

Error in Lagrange's method,

$$E = \max \left| \frac{f^{n+1}(x)}{(n+1)!} (x-x_0)(x-x_1)(x-x_2) \right|$$

where $E = |f(x) - p(x)|$

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$$\Rightarrow E = \max \left| \frac{f^3(x)}{3!} (x-0)(x-1/2)(x-1) \right|$$

$$\Rightarrow E = \max \left| \frac{f^3(x)}{3!} \left(x^3 - \frac{3x^2}{2} + \frac{x}{2} \right) \right|$$

Now max value of $|f^3(x)| = 1$ (given)

max value of $\left(x^3 - \frac{3x^2}{2} + \frac{x}{2} \right) \leq 1/2$

$$\therefore E = \max \left| \frac{f^3(x)}{6} \left(x^3 - \frac{3x^2}{2} + \frac{x}{2} \right) \right| \leq \frac{1}{6} \left(\frac{1}{2} \right)$$

$$\therefore E = |f(x) - p(x)| \leq \frac{1}{12}$$

Hence proved.

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6. (c) Consider the integers 13, 8 and their 8 bit binary representations. Perform the following bitwise operations and express the results in decimal system:

(i) 13 & 20 (bitwise AND)

(ii) 13|20 (bitwise OR)

(iii) 13^20 (bitwise XOR)

(iv) ~20 (bitwise Compliment) (15)

Calculating binary forms of 13, 20:

$2 \mid 13$, $2 \mid 6$ r 1, $2 \mid 3$ r 0, $2 \mid 1$ r 1, 0 r 1

$\Rightarrow 13 = (1101)_2$

$2 \mid 20$, $2 \mid 10$ r 0, $2 \mid 5$ r 0, $2 \mid 2$ r 1, $2 \mid 1$ r 0, 0 r 1

$\Rightarrow 20 = (10100)_2$

(i) 13&20 :

$$\begin{array}{r} 01101 \\ 10100 \\ \hline 00100 \end{array} = (00100)_2 = 4$$

$\therefore$ 13&20 = 4

(ii) 13|20 :

$$\begin{array}{r} 01101 \\ 10100 \\ \hline 11101 \end{array} = (11101)_2 = 29$$

$\therefore$ 13|20 = 29

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(iii) $13 \wedge 20$:

XOR TABLE		
		$A\bar{B} + \bar{A}B$
1	0	1
1	1	0
0	1	1
0	0	0

$$\begin{array}{r} 0\ 1\ 1\ 0\ 1 \\ 1\ 0\ 1\ 0\ 0 \\ \hline 1\ 1\ 0\ 0\ 1 \end{array} = (11001)_2 = 25$$

$\therefore 13 \wedge 20 = 25$

(iv) ~ 20 :

$$\sim\ 1\ 0\ 1\ 0\ 0$$

$$0\ 1\ 0\ 1\ 1 = (1011)_2 = 11$$

$\therefore \sim 20 = 11$

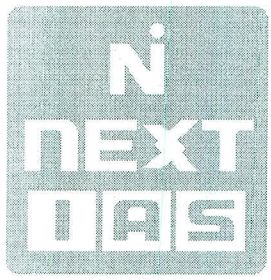
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7. (a) Solve the PDE $\left(D^2-D'^2+D+3D'-2\right)z=e^{x-y}-x^2y$. (15)



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7. (b) Evaluate $\int_0^1 e^{-x^2}\,dx$; employing three-point Gauss Quadrature formula, finding the required weights and residues. Use five decimal places for computation. (15)



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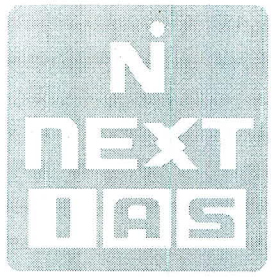
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7. (c) A particle of unit mass is projected so that its total energy is h in a field of force of which the potential energy is $\phi(r)$ at a distance r from the origin. By employing the principle of energy and least action, show that the path is given by $c^2\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]=r^4\left[h-\phi(r)\right]$, where c is constant. (20)

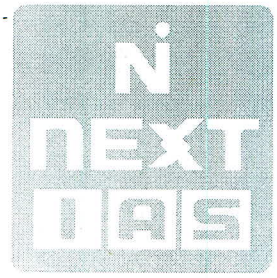


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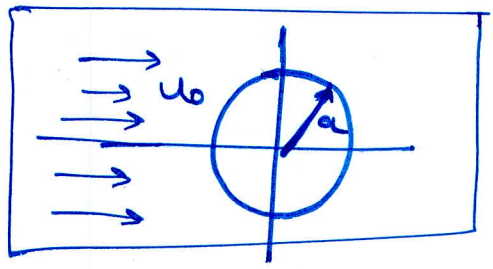
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8. (a) Consider a uniform flow U_0 in the positive x-direction. A cylinder of radius a is located at the origin. Find the stream function and the velocity potential. Find also the stagnation points. (15)

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The complex potential for the system is

$$w = U_0\left(z + \frac{a^2}{z}\right)$$. Let $z = x+iy$

$$\Rightarrow w = U_0\left((x+iy) + \frac{a^2}{x+iy} \times \frac{x-iy}{x-iy}\right)$$

$$\Rightarrow w = U_0\left((x+iy) + \frac{a^2}{x^2+y^2}(x-iy)\right)$$

$$\Rightarrow w = U_0\left(\left[x + \frac{a^2x}{x^2+y^2}\right] + i\left[y - \frac{a^2y}{x^2+y^2}\right]\right)$$

Considering the form $w = \phi + i\psi$,

$\phi = U_0\left(x + \frac{a^2x}{x^2+y^2}\right)$ = velocity potential

$\psi = U_0\left(y - \frac{a^2y}{x^2+y^2}\right)$ = stream function.

For the stagnation points, $q = 0$

Now

$$q_x = -\frac{\partial\phi}{\partial x}; \quad q_y = -\frac{\partial\phi}{\partial y}$$

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$\frac{\partial \phi}{\partial x} = 0 \Rightarrow 1 + a^2 \left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right] = 0.$

$\Rightarrow 1 + a^2 \frac{(y^2-x^2)}{(x^2+y^2)^2} = 0 \quad \text{—— (1)}.$

$\frac{\partial \phi}{\partial y} = 0 \Rightarrow a^2 x \left(\frac{-2ya^2}{(x^2+y^2)^2}\right) = 0 \quad \text{— (2)}.$

From (2), $x^2 y = 0$

$\Rightarrow x, y \to \infty$

$\therefore$ stagnation points located at ∞.

Let $y = 0$.

Then from (1), $1 = \frac{a^2 x^2}{x^4} \Rightarrow x = \pm a.$

Let $x = 0$

Then from (1), $1 + \frac{a^2 y^2}{y^4} = 0 \Rightarrow y = \pm ia.$

$\therefore$ stagnation points located at $(\pm a, 0)$ & $(0, \pm a)$

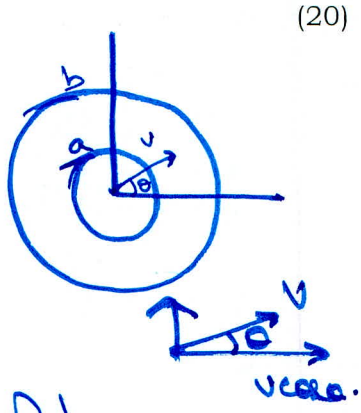
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8. (b) The space between two infinitely long coaxial cylinders of radii a and b $(b>a)$ respectively is filled by homogeneous fluid, of density ρ. The inner cylinder is suddenly moved with velocity v perpendicular to this axis, the outer being kept fixed. Show that the resulting impulsive pressure on a length l of inner cylinder is, $\pi\rho a^2 l\frac{b^2+a^2}{b^2-a^2}v$. (20)



In the system, the velocity potential ϕ is

$$\phi = \left(Ar+\frac{B}{r}\right)\cos\theta + \left(Cr+\frac{D}{r}\right)\sin\theta .$$

Now the fluid portion adjacent to inner cylinder also moves with velocity $v\cos\theta$

$$q = -\nabla\phi = -\frac{\partial\phi}{\partial r}$$

$$\Rightarrow \frac{\partial\phi}{\partial r} = \left(A-\frac{B}{r^2}\right)\cos\theta + \left(C-\frac{D}{r^2}\right)\sin\theta$$

$$q = v\cos\theta = \left(\frac{B}{r^2}-A\right)\cos\theta + \left(\frac{D}{r^2}-C\right)\sin\theta\Big|_{r=a}$$

$$\Rightarrow v\cos\theta = \left(\frac{B}{a^2}-A\right)\cos\theta + \left(\frac{D}{a^2}-C\right)\sin\theta \quad -(1)$$

At $r=b$, $V=0$

$$\Rightarrow 0 = \left(\frac{B}{b^2}-A\right)\cos\theta + \left(\frac{D}{b^2}-C\right)\sin\theta \quad -(2).$$

From (1) & (2), $C=D=0$

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Also $\frac{B}{a^2} - A = V$; $\frac{B}{b^2} - A = 0$

$\Rightarrow A = \frac{B}{b^2}$

$\Rightarrow \frac{B}{a^2} - \frac{B}{b^2} = V \Rightarrow \frac{B(b^2-a^2)}{a^2b^2} = V$

$\Rightarrow B = \frac{a^2b^2V}{b^2-a^2}$, $A = \frac{B}{b^2} = \frac{a^2V}{b^2-a^2}$

$\therefore \phi = \left(Ar + \frac{B}{r}\right)\cos\theta$

$\Rightarrow \phi = \left(\frac{a^2Vr}{b^2-a^2} + \frac{a^2b^2V}{r(b^2-a^2)}\right)\cos\theta$

$\phi = \frac{a^2V}{b^2-a^2}\left(r + \frac{b^2}{r}\right)\cos\theta$. At $r = a$,

$\phi = \frac{a^2V}{b^2-a^2}\frac{(a^2+b^2)}{a}\cos\theta = \frac{aV(a^2+b^2)\cos\theta}{b^2-a^2}$.

The pressure on the inner cylinder is given by

$\boxed{\rho\phi = P}$

$\Rightarrow P = \rho aV\frac{(a^2+b^2)}{(b^2-a^2)}\cos\theta$.

Along a strip of length l, inclined at θ angle to the x-axis;

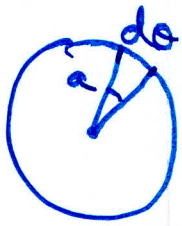
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$P' = P l \cos\theta$

$\Rightarrow P' = \rho a v l \frac{(a^2+b^2)}{(b^2-a^2)} \cos^2\theta$

Now the Impulsive force along a differential part is $dP_I = P' \times a\, d\theta$



$\therefore dP_I = \rho a^2 v l \frac{(a^2+b^2)}{b^2-a^2} \cos^2\theta\, d\theta.$

Integrating from $\theta = 0$ to $\theta = 2\pi$,

$P_I = \int dP_I = \int_0^{2\pi} \frac{\rho a^2 v l (a^2+b^2)}{(b^2-a^2)} \cos^2\theta\, d\theta$

$P_I = \frac{\rho \pi a^2 v l (a^2+b^2)}{b^2-a^2}$ $\left(\because \int_0^{2\pi} \cos^2\theta = \pi\right)$

$\therefore$ Impulsive Pressure $= \pi \rho a^2 l \frac{b^2+a^2}{b^2-a^2} v$

Hence proved.

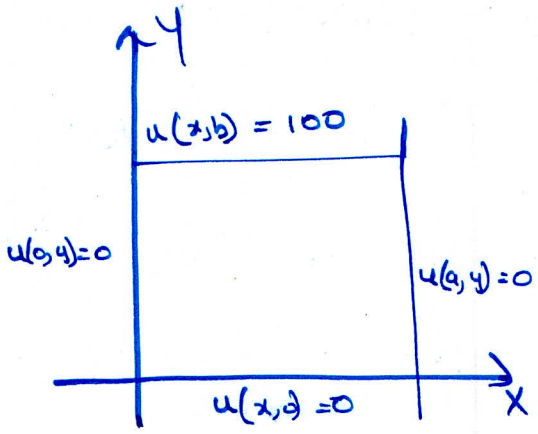
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8. (c) Find the steady state temperature distribution in a thin rectangular plate bounded by the lines $x=0, x=a, y=0$ and $y=b$. The edges $x=0, x=a$ and $y=0$ are kept at temperature zero while the edge $y = b$ is kept at 100°. (15)



To find steady state temp. distribution

⇒ Let $u(x,y)$ be the distribution

Then $\nabla^2 u = 0$

(or) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ... (1)

Let $u = X(x)\,Y(y)$

From (1); $X''Y + XY'' = 0$.

$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \boxed{\frac{X''}{X} = \frac{-Y''}{Y} = k}$

Case (i): Let $k=0 \Rightarrow X = a'x+b'$ and $Y = c'y+d'$

$u(x,y) = (a'x+b')(c'y+d')$

$u(x,0) = 0$, $u(0,y) = 0 \Rightarrow b' = d' = 0$.

$u(a,y) = 0 \Rightarrow a' = 0 \Rightarrow \boxed{u = 0}\ \forall (x,y)$

(contradiction.

Case II: Let $k = \lambda^2 \Rightarrow X'' - \lambda^2 X = 0$

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$\Rightarrow (D^2 - \lambda^2)X = 0 \Rightarrow X = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$

If λy $Y'' + \lambda^2 Y = 0 \Rightarrow (D^2 + \lambda^2) Y = 0$

$\Rightarrow Y = C_3 \sin \lambda y + C_4 \cos \lambda y$

$\Rightarrow u = (C_1 e^{\lambda x} + C_2 e^{-\lambda x})(C_3 \sin \lambda y + C_4 \cos \lambda y)$

$u(x,0) = 0 \Rightarrow C_4 = 0$; $u(0,y) = 0 \Rightarrow C_4 + C_2 = 0$.

$u(a,y) = 0 \Rightarrow C_3 = 0$

$C_4 = 0$; $C_3 = 0 \Rightarrow u = 0 \ \forall (x,y)$ contradiction.

(iv) Taking $k = -\lambda^2$

$\Rightarrow (D^2 + \lambda^2)X = 0 \Rightarrow X = C_1 \cos \lambda x + C_2 \sin \lambda x$

Also $(D^2 - \lambda^2) Y = 0 \Rightarrow Y = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$.

$\therefore u = XY = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{\lambda y} + C_4 e^{-\lambda y})$

$u(x,0) = 0 \Rightarrow C_3 + C_4 = 0 \Rightarrow C_3 = -C_4$

$u(0,y) = 0 \Rightarrow C_1 = 0$.

$\therefore u = (C_2 \sin \lambda x)(C_3 e^{\lambda y} + C_4 e^{-\lambda y})$

$u(a,y) = 0 \Rightarrow \sin \lambda a = 0 \Rightarrow \lambda a = n\pi$

$\Rightarrow \lambda = \frac{n\pi}{a}$

$\therefore u = \sum_{n=1}^{\infty} \left(C_2 \sin \frac{n\pi x}{a}\right)\left(C_3 e^{\frac{n\pi y}{a}} + C_4 e^{-\frac{n\pi y}{a}}\right)$

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Also $u(x,b) = 100$

$$\Rightarrow 100 = \sum_{n=1}^{\infty} C_2 C_4 \sin\left(\frac{n\pi x}{a}\right)\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right)$$

$$\Rightarrow C_2 C_4 = \frac{2}{a}\int_0^a 100 \sin\left(\frac{n\pi x}{a}\right) dx.$$

$$\Rightarrow C_2 C_4 = \frac{200}{a}\int_0^a \sin\frac{n\pi x}{a}\, dx = \frac{200}{a}\left(-\frac{\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)_0^a$$

$$\Rightarrow \frac{200}{a} \times \frac{a}{n\pi}\left(\cos\frac{n\pi x}{a}\right)_a^0 = \frac{200}{n\pi}(1-(-1)^n)$$

$n \neq$ even no of $C_2 C_4 \neq 0$

$$\therefore C_2 C_4 = \frac{400}{(2m+1)\pi} \text{ where } n = 2m+1$$

$$\therefore u(x,y) = \sum_{m=0}^{\infty} \frac{400}{(2m+1)\pi} \sin\left(\frac{n\pi x}{a}\right)\left[e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right]$$

where $n = 2m+1$

$$\Rightarrow \boxed{u(x,y) = \sum_{m=0}^{\infty} \frac{400}{(2m+1)\pi} \sin\left(\frac{(2m+1)\pi x}{a}\right)\left[e^{\frac{(2m+1)\pi y}{a}} - e^{-\frac{(2m+1)\pi y}{a}}\right]}$$

is the reqd temp distribution at steady state.

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SPACE FOR ROUGH WORK

$\frac{1+i}{1-i} \times \frac{1-i}{1-i} = \frac{1-1}{1+1}$

$(xh)^2 k (x+h)^2$

$xhx^{-1}x^{-1}x$

$x^{-1}x \cdot x\, hx^{-1}x^{-1}x$

$x^{-1}x^2\, h\, x^{-2}x$

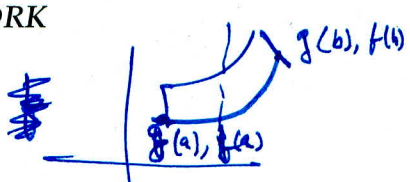
$h^2 \in H$

$h^2 \cdot h^2 \in H$

$h'(c)$

$x\, x\, h^2 \cdot h^2 x^{-1}x^{-1}$

$(xh)^2 \cdot h' \cdot (x+h)^2$



$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$k(x) + I = p(x) + I$.

$k(x) - p(x) \in I$

$g(x) \in I$

0.0000003
0.0000002

$g(x)f'(x)\ (x-1)(x^2-7x+12)$

$x^3-7x^2+12x-x^2+7x-12$

$x^3-8x^2+19x-12$.

$= k_1(a+ib)(a-ib)$

$(1+i) = k_1(a^2+b^2)$

$(1+i) = k_2(a+ib)(c+id)$

$\Rightarrow 1 = \frac{k_1}{k_2} \frac{(a-ib)}{(c+id)}$

