

NEXT IAS

(To be filled by candidate)

Name of Candidate : MADHUKANT

Roll No. :

Registration Number :

Date of Examination : 05/07/24

Exam Centre : Old Rajinder Nagar ☐

Online ☐

CSE (MAINS) TEST SERIES - 2024

Test-01

MATHEMATICS
OPTIONAL

Dated : 23-06-2024

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are **EIGHT** questions divided in **Two** sections and printed in **ENGLISH**.

Candidate has to attempt **FIVE** questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	20			
2. (b)	18	15			
2. (c)	21	15			
3. (a)	24	15			
3. (b)	27	15			
3. (c)	30	20			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	40	20			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	15			
6. (b)	57	20			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	15			
7. (c)	70	20			
8. (a)	74	15			
8. (b)	77	20			
8. (c)	81	15			
Grand Total					

Remarks

Observations:

SECTION-A

Q.1 (a)

If V is finite dimensional and f is a homomorphism of V (onto) V , prove that f must be one one and so, an isomorphism.

(10)

Let V be finite dimensional Vector space
and $f: V \rightarrow V$ be homomorphism
and
onto

To prove: f is one-one or Isomorphism.

Let $\dim(V) = n$ and basis of V be -

$$S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}.$$

Now, $\because f$ is onto $\therefore \forall \beta \in S, \exists \alpha \in V$ s.t.
 $f(\alpha) = \beta.$

Let $a, b \in V$ s.t. $f(a) = \alpha_1$ and $f(b) = \alpha_2.$

claim: a, b are Linearly Independent

Proof: Consider, $xa + yb = 0$

$$f(xa + yb) = f(0) \quad \text{--- (1)}$$

and $\because f$ is homomorphism i.e. $f(a+b) = f(a) + f(b)$

$$\Rightarrow f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$$

$\Rightarrow$ eqn (1) Can be written as $x f(a) + y f(b) = 0$

$$x\alpha_1 + y\alpha_2 = 0$$

$$\therefore \alpha_1, \alpha_2 \text{ are L.I.} \Rightarrow x = y = 0$$

$\therefore$ pre-image of $\alpha_1, \alpha_2, \dots, \alpha_n$ are L.I.

Now, Consider, any $\alpha, \beta \in V$ s.t $\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$
 $f(\alpha) = f(\beta)$ & $\beta = b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n$

$$\Rightarrow f(a_1\alpha_1 + \dots + a_n\alpha_n) = f(b_1\alpha_1 + \dots + b_n\alpha_n)$$

$$\Rightarrow (a_1 - b_1)f(\alpha_1) + (a_2 - b_2)f(\alpha_2) + \dots + (a_n - b_n)f(\alpha_n) = 0$$

Q.1 (b) Determine the set of points where the function $f(x) = \frac{x}{1+|x|}$ is differentiable. (10)

Consider $f(x) = \frac{x}{1+|x|}$

$\therefore 1+|x| \geq 1 \Rightarrow \frac{1}{1+|x|}$ exist $\forall x \in \mathbb{R}$

and $f(x)$ is continuous $\forall x \in \mathbb{R}$.

$$f(x) = \begin{cases} \frac{x}{1+x} & x \geq 0 \\ \frac{x}{1-x} & x < 0 \end{cases}$$

Now, Clearly, $f(x)$ is differentiable for $x > 0$
and for $x < 0$

At $x=0$,

$$LHD = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \frac{-h}{1-h} = \frac{1}{1-h} \approx 1$$

$$RHD = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \frac{h}{1+h} = \frac{1}{1+h} \approx 1$$

$$\Rightarrow LHD = RHD$$

$\therefore$ f is differential $\forall x \in \mathbb{R}$

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Q.1 (c) Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \cot^2 x \right)$

(10)

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$$\text{let } L = \lim_{x \rightarrow 0} \frac{1}{x^2} - \cot^2 x$$

$$= \lim_{x \rightarrow 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{(\tan x - x)(\tan x + x)}{x^2 \tan^2 x}$$

$$\text{Wkt, } \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} - \dots$$

$$\therefore L = \lim_{x \rightarrow 0} \frac{\left(\frac{x^3}{3} + \frac{2x^5}{15} - \dots \right) (\tan x + x)}{x^2 \tan^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{x^3 \left(\frac{1}{3} + \frac{2x^2}{15} + \dots \right) (\tan x + x)}{x^2 \tan^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{\left(\frac{1}{3} + \dots \right) \left(\frac{\tan x}{x} + 1 \right)}{\left(\frac{\tan^2 x}{x^2} \right)}$$

And, we know that $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

$$\therefore L = \lim_{x \rightarrow 0} \frac{\left(\frac{1}{3} + \frac{2x^2}{15} + \dots \right) \left(\frac{\tan x}{x} + 1 \right)}{\left(\frac{\tan^2 x}{x^2} \right)}$$

$$= (1+1) \left(\frac{1}{3} \right) = \boxed{\frac{2}{3}}$$

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Q.1 (d) Find the centre of the conic $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1, 2x + 2y - z = 3$. (10)

Given Ellipsoid $E: \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1$

and plane : $P: 2x + 2y - z = 3$

Intersection of above will form section of plane

Now, wkt, if locus of line intersecting with ellipsoid having mid point at (x, y, z)

is given by $T = S_1$

this plane will be the given plane in question (P)

$\therefore$ plane with mid point (x, y, z) will be

$$\frac{xx_1}{9} + \frac{yy_1}{16} + \frac{zz_1}{4} - 1 = \frac{x_1^2}{9} + \frac{y_1^2}{16} + \frac{z_1^2}{4} - 1$$

$$\frac{xx_1}{9} + \frac{yy_1}{16} + \frac{zz_1}{4} = \frac{x_1^2}{9} + \frac{y_1^2}{16} + \frac{z_1^2}{4}$$

On comparing with $2x + 2y - z = 3$

$$\frac{x/9}{2} = \frac{y/16}{2} = \frac{z/4}{-1} = \frac{\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4}}{3} = \lambda(\text{say})$$

$$\therefore x_1 = 18\lambda, y_1 = 32\lambda, z_1 = -4\lambda$$

$$& \quad x_1^2/9 + y_1^2/16 + z_1^2/4 = 3\lambda$$

$$\frac{18 \cdot 18}{9} \lambda^2 + \frac{32 \cdot 32}{16} \lambda^2 + \frac{4 \cdot 4}{4} \lambda^2 = 3\lambda$$

$$\lambda(36\lambda + 64\lambda + 4\lambda - 3) = 0$$

$$\lambda(104\lambda - 3) = 0 \quad \lambda = \frac{3}{104} \quad \text{or } \lambda = 0$$

$\therefore$ Plane does not pass through 0 $\therefore \lambda = 0$ is not feasible

$$\text{for } \lambda = \frac{3}{104}$$

Centre of Conic will be

$$(x_1, y_1, z_1) = \left(18 \times \frac{3}{104}, 32 \times \frac{3}{104}, -4 \times \frac{3}{104} \right)$$

$$= \left(\frac{27}{52}, \frac{24}{26}, -\frac{3}{26} \right)$$

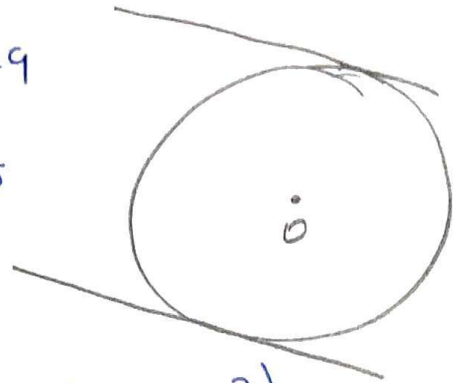
$$= \boxed{\left(\frac{27}{52}, \frac{12}{13}, -\frac{3}{26} \right)}$$

Q.1 (e)

Find the equation of the cylinder whose generators touch the sphere $x^2 + y^2 + z^2 = 9$ and are perpendicular to the plane $x - y - 3z = 5$. (10)

Cylinder whose generator touches the sphere $x^2 + y^2 + z^2 = 9$ and $\perp$ to plane $x - y - 3z = 5$

is enveloping cylinder of Sphere whose d.c. are $\parallel$ to $(1, -1, -3)$



eqⁿ of enveloping cylinder is given by $\rightarrow$

$$Ss_1 = t^2, \text{ where } S_1 \equiv l^2 + m^2 + n^2 - 9 = 0$$

$$t \equiv xl + ym + zn = 0$$

$\therefore (l, m, n)$ are d.c.

$$\therefore (l, m, n) = \left(\frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{-3}{\sqrt{11}} \right)$$

$$\therefore (x^2 + y^2 + z^2 - 9) \left(x^2 \frac{1}{11} + \frac{1}{11} + \frac{9}{11} \right) = \frac{(x - y - 3z)^2}{11}$$

$$11(x^2 + y^2 + z^2 - 9) = (x - y - 3z)^2$$

$$\boxed{10x^2 + 10y^2 + 2z^2 + 2xy + 6xz + 6yz - 99 = 0}$$

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Q.2 (a) Let V be vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let V_e be the set of all even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions, $f(-x) = -f(x)$.

i. Prove that V_e and V_o are subspaces of V

ii. Prove that $V_e + V_o = V$

iii. Prove that $V_e \cap V_o = \{0\}$.

(20)

Given: V be vector space of all function $\mathbb{R} \rightarrow \mathbb{R}$

V_e be set of even function i.e. $f \in V_e \Rightarrow f(-x) = f(x)$

V_o be set of odd function i.e. $f \in V_o \Rightarrow f(-x) = -f(x)$.

(i) To prove: V_e, V_o are subspace of V

let $f_1, f_2 \in V_e$ and $a, b \in \mathbb{R}$

i.e. $f_1(-x) = f_1(x)$ and $f_2(-x) = f_2(x)$

then, $(af_1 + bf_2)(-x) = (af_1)(-x) + (bf_2)(-x)$

$$= af_1(-x) + bf_2(-x), (\because f_1, f_2 \in V)$$

$$= af_1(x) + bf_2(x) \quad \& \text{ follow property of vector space}$$

$$= (af_1 + bf_2)(x)$$

$$\Rightarrow af_1 + bf_2 \text{ is even} \Rightarrow af_1 + bf_2 \in V_e$$

$$\Rightarrow \boxed{V_e \text{ is Subspace of } V}$$

Similarly, let $f_1, f_2 \in V_o$ and $a, b \in \mathbb{R}$

$$\Rightarrow f_1(-x) = -f_1(x) \text{ and } f_2(-x) = -f_2(x)$$

$$\begin{aligned} \text{then } (af_1 + bf_2)(-x) &= a f_1(-x) + b f_2(-x) \\ &= - (af_1(x) + bf_2(x)) \\ &= - (af_1 + bf_2)(x) \end{aligned}$$

$\Rightarrow af_1 + bf_2$ is odd function

$$\Rightarrow \boxed{V_o \text{ is Subspace of } V}$$

$$(ii) \quad V_e + V_o = V$$

$$V_e + V_o = \{f_1 + f_2 \mid f_1 \in V_e, f_2 \in V_o\}$$

$$\therefore V_e, V_o \subseteq V$$

$$\Rightarrow \underline{V_e + V_o \subseteq V} \quad \text{--- (1)}$$

Let $f \in V$ be any function.

We know by Taylor's expansion f can be expanded as—

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \infty$$

On segregating odd and even terms.

$$f(x) = (a_0 + a_2x^2 + a_4x^4 + \dots) + (a_1x + a_3x^3 + \dots)$$

$$\text{clearly } (a_0 + a_2x^2 + \dots) \in V_e$$

$$\text{and } (a_1x + a_3x^3 + \dots) \in V_o$$

$$\therefore f \text{ can be written as } f = g_1 + g_2$$

$$\text{where } g_1(x) = a_0 + a_2x^2 + \dots$$

$$g_2(x) = a_1x + a_3x^3 + \dots$$

$$\therefore f \in V_e + V_o$$

$$\Rightarrow V \subseteq V_e + V_o \quad \text{--- (2)}$$

$\therefore$ From eqⁿ (1) and (2)

$$\boxed{V = V_e + V_o}$$

(iii) $V_e \cap V_o = \{0\}$

Let any function f s.t

$$f \in V_e \text{ and } f \in V_o$$

$$\therefore f(-x) = f(x) \text{ and } f(-x) = -f(x)$$

But, $\therefore$ Both will be equal

$$\therefore f(x) = -f(x)$$

$$\Rightarrow f(x) = 0$$

$$\Rightarrow \text{only } 0 \in V_e \cap V_o$$

$$\therefore \boxed{V_e \cap V_o = \{0\}}$$

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Q.2 (b) Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$ for $0 < x < \frac{\pi}{2}$.

(15)

To show, $\frac{\tan x}{x} > \frac{x}{\sin x} \quad \forall \quad 0 < x < \frac{\pi}{2}$.

or $\tan x \cdot \sin x - x^2 > 0$

let $f(x) = \tan x \cdot \sin x - x^2$, $f(0) = 0$

$$\begin{aligned} f'(x) &= \sec^2 x \cdot \sin x + \tan x \cdot \cos x - 2x \\ &= \tan x \cdot \sec x + \sin x - 2x \end{aligned}$$

$f'(0) = 0$.

$$\begin{aligned} f''(x) &= \sec^3 x + \sec x \cdot \tan^2 x + \cos x - 2 \\ &= \sec x (\tan^2 x + 1) + \sec x \cdot \tan^2 x + \cos x - 2 \\ &= 2 \cdot \sec x \cdot \tan^2 x + \sec x + \cos x - 2 \\ &= 2 \cdot \sec x \cdot \tan^2 x + (\sqrt{\cos x} - \sqrt{\sec x})^2 \end{aligned}$$

Now, In $x \in (0, \frac{\pi}{2})$ $\sec x > 0$ and $\tan x > 0$

and $(\sqrt{\cos x} - \sqrt{\sec x})^2 > 0$

$\Rightarrow f''(x) > 0$ in $x \in (0, \frac{\pi}{2})$

$\Rightarrow f'(x)$ is increasing function in $(0, \frac{\pi}{2})$

$\Rightarrow f'(x) > f'(0)$ in $(0, \frac{\pi}{2})$

$\Rightarrow f(x) > 0$

$\Rightarrow f(x)$ is increasing function

$\Rightarrow f(x) > f(0)$ in $x \in (0, \frac{\pi}{2})$

$\Rightarrow f(x) > 0$ in $x \in (0, \frac{\pi}{2})$

$\Rightarrow \tan x \cdot \sin x - x^2 > 0$

$\Rightarrow \boxed{\frac{\tan x}{x} > \frac{x}{\sin x} \text{ in } x \in (0, \frac{\pi}{2})}$

Q.2 (c)

Find the equation of two straight lines through the point $(1, 1, 1)$ that intersect the line $x-4=2(y-4)=2(z-1)$ at an angle of 60° .

(15)

let given line be

$$L: \frac{x-4}{1} = \frac{y-4}{\frac{1}{2}} = \frac{z-1}{\frac{1}{2}}$$

and point be $A(1, 1, 1)$

We need to find line passing through A that intersect L at angle 60° .

let required line be $\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-1}{n}$

let line intersect at P

and P can be written as in form of lying on line L

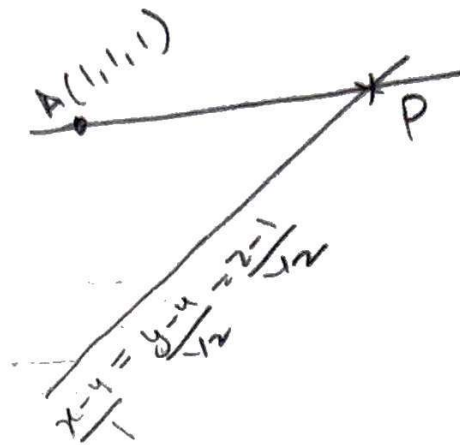
$$P(t+4, \frac{1}{2}t+4, \frac{1}{2}t+1)$$

d.r of AP is given by $(t+3, \frac{t}{2}+3, \frac{t}{2})$

if AP forms 60° with Line L then -

$$\cos 60^\circ = \frac{1}{2} = \frac{(t+3) + (\frac{t}{2}+3)\frac{1}{2} + \frac{t}{2} \cdot \frac{1}{2}}{\sqrt{1^2 + \frac{1}{2^2} + \frac{1}{2^2}} \cdot \sqrt{(t+3)^2 + (\frac{t}{2}+3)^2 + (\frac{t}{2})^2}}$$

$$\frac{1}{2} = \frac{1}{4} \frac{(4t+12 + t+6 + t)}{\sqrt{\frac{3}{2} \cdot \frac{1}{2}} \sqrt{4(t+3)^2 + (t+6)^2 + t^2}}$$



$$\Rightarrow 2 = \frac{(6t + 18) \times 2\sqrt{2}}{\sqrt{3} \sqrt{4(t^2 + 9 + 6t) + t^2 + 36 + 12t + t^2}}$$

$$1 = \frac{6(t+3) \times \sqrt{2}}{\sqrt{3} (6t^2 + 36t + 72)^{1/2}}$$

$$1 = \frac{6\sqrt{2} \sqrt{2} (t+3)}{\sqrt{2}\sqrt{3} \sqrt{6} (t^2 + 6t + 12)^{1/2}}$$

$$(t^2 + 6t + 12)^{1/2} = 2(t+3)$$

$$t^2 + 6t + 12 = 4(t^2 + 9 + 6t)$$

$$3t^2 + 18t + 24t = 0$$

$$t^2 + 6t + 8t = 0$$

$$(t+4)(t+2) = 0$$

$$\Rightarrow t = -2, t = -4$$

$\therefore$ P will be $(2, 3, 0)$

and $(0, 2, -1)$

$\therefore$ two lines will be formed by

$A(1, 1, 1)$ and $(2, 3, 0)$ & $(1, 1, 1)$ and $(0, 2, -1)$

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{-1}$$

$$\frac{x-1}{-1} = \frac{y-1}{1} = \frac{z-1}{-2}$$

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- Q.4 (a) Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 4 \\ 1 & -2 & 2 \end{bmatrix}$. Find the matrix B that represents the linear operator A relative to the basis $S(u_1, u_2, u_3) = \{(1, 1, 0), (0, 1, 1), (1, 2, 2)\}$. (15)

Given that, A is linear operator relative to Standard Basis $B = \{e_1, e_2, e_3\}$
 $= \{(1, 0, 0), (0, 1, 1), (0, 0, 1)\}$

To find: matrix B that represent A relative to Basis $S(u_1, u_2, u_3) = \{(1, 1, 0), (0, 1, 1), (1, 2, 2)\}$

Transition Matrix from S to B

$$\begin{aligned} u_1 &= (1, 1, 0) = 1 \cdot e_1 + 1 \cdot e_2 + 0 \cdot e_3 \\ u_2 &= (0, 1, 1) = 0 \cdot e_1 + 1 \cdot e_2 + 1 \cdot e_3 \\ u_3 &= (1, 2, 2) = 1 \cdot e_1 + 2 \cdot e_2 + 2 \cdot e_3 \end{aligned}$$

transition matrix $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Now, linear transformation A corresponding to Basis S will be

$$B = P^{-1}A$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 4 \\ 1 & -2 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 7 & 2 \\ 1 & 6 & 4 \\ 0 & -4 & -1 \end{bmatrix}$$

Q.4 (b)

Does the integral $\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}$ exist? if so, find its value.

(15)

Consider, $I = \int_{-1}^1 \sqrt{\frac{1+x}{1-x}} dx$

Let $f(x) = \sqrt{\frac{1+x}{1-x}}$

clearly f has ∞ discontinuity only at $x=1$.

Now, consider $g(x) = \frac{1}{\sqrt{1-x}}$

$\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 1} \sqrt{1+x} = \sqrt{2}$ (finite no)

$\therefore$ By Comparison test, $\int_{-1}^1 f(x) dx$ and

$\int_{-1}^1 g(x) dx$ Converges or diverges together

and $\int_{-1}^1 g(x) dx = \int_{-1}^1 \frac{1}{\sqrt{1-x}} dx$ $n = \frac{1}{2} < 1$

$\therefore \int_{-1}^1 g(x) dx$ is Convergent

$\Rightarrow \int_{-1}^1 f(x) dx$ is Convergent.

$\therefore I = \int_{-1}^1 \sqrt{\frac{1+x}{1-x}} dx$ exist

Now, $I = \int_{-1}^1 \sqrt{\frac{1+x}{1-x}} dx$

let $x = \cos 2\theta$ $dx = -2 \sin 2\theta d\theta$

at $x = -1$, $\theta = \pi/2$ at $x = 1$, $\theta = 0$

$$\Rightarrow I = \int_{\pi/2}^0 \sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}} (-2 \sin 2\theta) d\theta$$

$$= \int_0^{\pi/2} \left| \frac{\cos \theta}{\sin \theta} \right| 2 \sin 2\theta d\theta$$

$$= 4 \int_0^{\pi/2} \cos^2 \theta d\theta$$

$$= \frac{4}{2} \int_0^{\pi/2} 1 + \cos 2\theta d\theta$$

$$= 2 \left(\frac{\pi}{2} + \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/2} \right)$$

$$I = \pi$$

$$\therefore \int_{-1}^1 \sqrt{\frac{1+x}{1-x}} dx = \pi$$

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Q.4 (c)

A point P moves on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, which is fixed. The plane through P and perpendicular to OP meets the axes in A, B, C respectively. The planes through A, B, C parallel to yz, zx and xy planes respectively intersect at Q. Prove that the locus of Q is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

(20)

Let P be any point on
plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

~~and P be P(x, y, z)~~

Let P be (x_1, y_1, z_1)

Now, plane through P

and $\perp$ to OP will be $\rightarrow$

$$x_1(x - x_1) + y_1(y - y_1) + z_1(z - z_1) = 0$$

$$xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2$$

Now, if above plane meet coordinate axes in A, B, C
then $A \left(\frac{x_1^2 + y_1^2 + z_1^2}{x_1}, 0, 0 \right)$

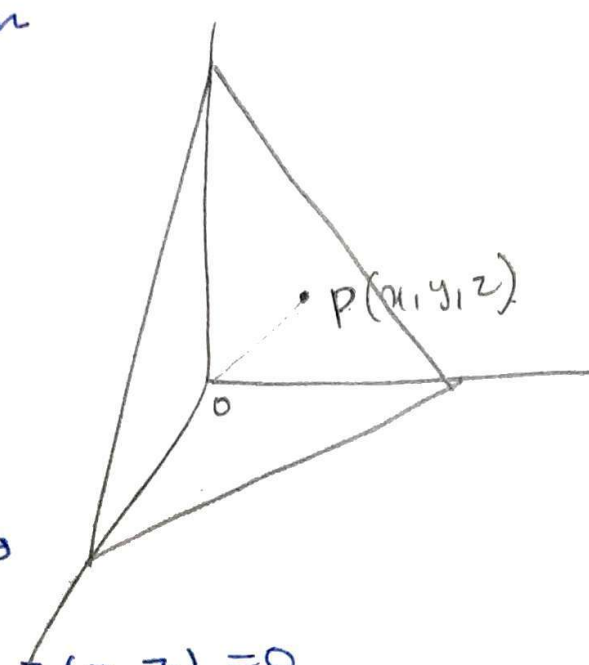
$$B \left(0, \frac{x_1^2 + y_1^2 + z_1^2}{y_1}, 0 \right)$$

$$C \left(0, 0, \frac{x_1^2 + y_1^2 + z_1^2}{z_1} \right)$$

Now planes through A and $\parallel$ to yz plane is

$$x = \frac{x_1^2 + y_1^2 + z_1^2}{x_1}$$

Similarly, other planes will be $\rightarrow$



$$y = \frac{x^2 + y^2 + z^2}{y}$$

$$z = \frac{x^2 + y^2 + z^2}{z}$$

Now point of intersection of above 3 planes
will give coordinate for Q (α, β, γ)

$$\therefore \text{let } (x, y, z) = \left(\frac{x^2 + y^2 + z^2}{x}, \frac{x^2 + y^2 + z^2}{y}, \frac{x^2 + y^2 + z^2}{z} \right)$$

$$\text{Now, } \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{(x^2 + y^2 + z^2)}$$

$$\text{Also, } \frac{1}{\alpha a} + \frac{1}{\beta b} + \frac{1}{\gamma c} = \frac{x/a + y/b + z/c}{x^2 + y^2 + z^2}$$

$$\therefore (x, y, z) \text{ lie on plane } x/a + y/b + z/c = 1$$

$$\therefore \frac{1}{a\alpha} + \frac{1}{b\beta} + \frac{1}{c\gamma} = \frac{1}{x^2 + y^2 + z^2}$$

$\therefore$ locus of Q can be given as.

$$\boxed{\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$$

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SECTION-B

- Q.5 (a) The eigenvalues of a real symmetric matrix A are $-1, 1$ and -2 . The corresponding eigenvectors are $\frac{1}{\sqrt{2}}(-1 \ 1 \ 0)^T$, $(0 \ 0 \ 1)^T$ and $\frac{1}{\sqrt{2}}(-1 \ -1 \ 0)^T$ respectively. Find the matrix A^4 . (10)

let $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = -2$ be eigenvalues of
real symmetric matrix A

and its corresponding eigen vectors be

$$X_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad X_3 = \begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

We know that, $\therefore A$ has all 3 distinct

eigen values, we can form diagonalisation of A

by $P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, where $P = [X_1 \ X_2 \ X_3]$
 $= D$ (say)

$$\therefore A = PDP^{-1}$$

$$A^4 = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1}) \\ = P D^4 P^{-1}$$

and $\therefore A^4 = \left\{ P \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}^4 P^{-1} \right.$
 $= P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix} P^{-1}$

$$A^4 = \begin{bmatrix} -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ +1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix} P^{-1}$$

$$A^4 = \begin{bmatrix} 17/2 & 15/2 & 0 \\ 15/2 & 17/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q.5 (b) Show that $\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} \log(1+a), a \geq 0$

(10)

To Show: $\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} \log(1+a), a > 0$

let $f(a) = \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)} dx$ — (1)

differentiating using Leibnitz Rule

$$f'(a) = \int_0^{\infty} \frac{1}{1+(ax)^2} \times x \times \frac{1}{x(1+x^2)} dx + 0 - 0$$

$$= \int_0^{\infty} \frac{1}{(1+a^2x^2)(1+x^2)} dx$$

$$= \frac{1}{(a^2-1)} \int_0^{\infty} \frac{a^2(1+x^2) - (1+a^2x^2)}{(1+a^2x^2)(1+x^2)} dx$$

$$= \frac{1}{a^2-1} \int_0^{\infty} \frac{a^2}{1+a^2x^2} - \frac{1}{1+x^2} dx$$

$$= \frac{1}{a^2-1} \left[\frac{a^2}{a} [\tan^{-1}(ax)]_0^{\infty} - [\tan^{-1}x]_0^{\infty} \right]$$

$$= \frac{1}{a^2-1} [a \cdot \frac{\pi}{2} - \frac{\pi}{2}]$$

$$= \frac{\pi}{2} \frac{1}{a+1}$$

On Integrating,

$$f(a) = \int \frac{\frac{\pi}{2}}{a+1} da + C$$

$$= \frac{\pi}{2} \log(a+1) + C$$

at $a=0$ from eq (1)

$$f(0) = \int_0^{\infty} 0 \cdot dx = 0$$

$$\Rightarrow f(a) = \frac{\pi}{2} \log(a+1) + 0$$

$$\boxed{f \Rightarrow \int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} \log(1+a)}$$

Q.5 (c)

Prove that the function $f(x, y) = \sqrt{|xy|}$ is not differentiable at the point $(0, 0)$, but f_x and f_y both exist at the origin and have the value 0.

(10)

Given function:

$$f(x, y) = \sqrt{|xy|}$$

To prove: ① f_x, f_y exist at $(0, 0)$

② $f_x = f_y = 0$ at $(0, 0)$

③ f is not differentiable at $(0, 0)$

$f_x \rightarrow$

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{|h| \cdot 0} - 0}{h} = 0$$

$$f_y(0, 0) = \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} = \frac{0 - 0}{k} = 0$$

$\therefore f_x, f_y$ exist at $(0, 0)$

and $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$

differentiability of f at $(0, 0)$

$$\frac{df}{dt} f'(0, 0) = \lim_{h, k \rightarrow 0, 0} \frac{f(h, k) - f(0, 0)}{\sqrt{h^2 + k^2}}$$

$$= \lim_{h,k \rightarrow 0,0} \frac{\sqrt{|h||k|}}{\sqrt{h^2+k^2}}$$

(taking limit along $k=mh$ line)

$$= \lim_{h \rightarrow 0} \frac{\sqrt{m} |h|}{\sqrt{1+m^2} |h|} = \sqrt{\frac{m}{1+m^2}}$$

this value of limit depend on value of m
and hence limit do not exist

$\Rightarrow f$ is not differentiable at $(0,0)$

Q.5 (d)

Prove that the equations of the chord through the point $(1, 2, 3)$ which is bisected by the diametral plane $10x - 24y = 21$ of the paraboloid $5x^2 - 6y^2 = 7z$ are $(x-1) = \frac{1}{2}(y-2) = \frac{1}{3}(z-3)$.

let ~~the~~ chord passes through

$A(1, 2, 3)$ and is bisected

by plane $10x - 24y = 21$ at M .

Given Paraboloid $5x^2 - 6y^2 = 7z$

let equation of chord be

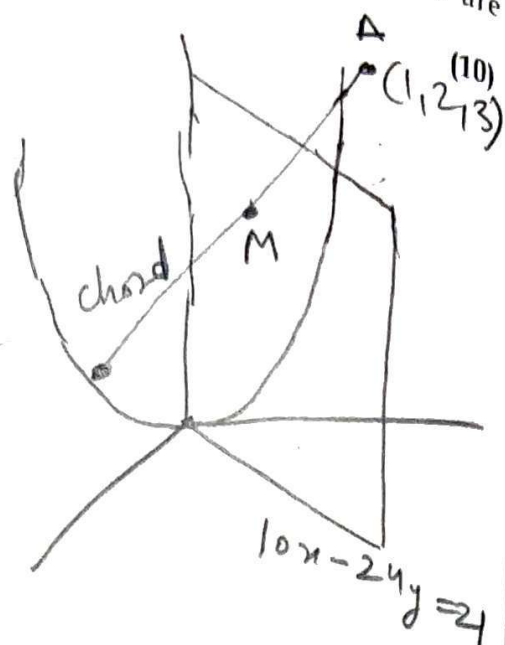
$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n}$$

Any point on chord will be $(lt+1, mt+2, nt+3)$

Intersection of chord with paraboloid $\rightarrow$

$$5(lt+1)^2 - 6(mt+2)^2 = 7(nt+3)$$

t^2



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- Q.5 (e) Find the equation of the tangent planes to the hyperboloid $2x^2 - 6y^2 + 3z^2 = 5$ which pass through the line $x + 9y - 3z = 0 = 3x - 3y + 6z - 5$. (10)

Given hyperboloid: $2x^2 - 6y^2 + 3z^2 = 5$

Given Line L : $x + 9y - 3z = 0 = 3x - 3y + 6z - 5$

Any plane passing through line L can be written as $\rightarrow$

$$(x + 9y - 3z) + \lambda(3x - 3y + 6z - 5) = 0$$

$$\text{i.e. } x(1 + 3\lambda) + y(9 - 3\lambda) + z(-3 + 6\lambda) - 5\lambda = 0$$

if the above plane is tangent to above hyperboloid then

$$d^2 = \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}, \text{ where } lx + my + nz = d \text{ is tangent plane}$$

and $ax^2 + by^2 + cz^2 = 1$ is conicoid

$$\therefore a = 2/5, \quad b = -6/5, \quad c = +3/5$$

$$\therefore 25d^2 = \frac{(3\lambda + 1)^2}{2/5} + \frac{(3\lambda - 9)^2}{-6/5} + \frac{(6\lambda - 3)^2}{3/5}$$

$$5d^2 = \frac{(3\lambda + 1)^2}{2} + \cancel{(\cancel{8\lambda - 3})^2} + \frac{9(\lambda - 3)^2}{-6} + \frac{9(2\lambda - 1)^2}{3}$$

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$$\Rightarrow 5\lambda^2 = \frac{(9\lambda^2 + 1 + 6\lambda)}{2} - \frac{3(\lambda^2 + 9 - 6\lambda)}{2} + 3(4\lambda^2 + 1 - 4\lambda)$$

$$= \frac{6\lambda^2 - 26 + 24\lambda}{2} + 12\lambda^2 + 3 - 12\lambda$$

$$= 3\lambda^2 + 12\lambda - 13 + 12\lambda^2 - 12\lambda + 3$$

$$10\lambda^2 - 10 = 0 \quad \lambda = \pm \frac{1}{\sqrt{10}}$$

$$= 3\lambda^2 - 13 + 12\lambda + 12\lambda^2 + 3 - 12\lambda$$

$$= 15\lambda^2 - 10$$

$$\Rightarrow \lambda^2 = 1 \Rightarrow \boxed{d = \pm 1}$$

$\Rightarrow$ eqⁿ of tangent planes will be $\rightarrow$

$$\boxed{4x + 6y + 3z - 5 = 0}$$

$$\boxed{-2x + 12y - 9z + 5 = 0}$$

Q.7 (a)

If S is a skew-Hermitian matrix, then show that $A = (I + S)(I - S)^{-1}$ is a unitary matrix. Also show that every unitary matrix can be expressed in the above form provided -1 is not an eigen value of A . (15)

Given that: S is skew-Hermitian Matrix

$$\Rightarrow \boxed{S^\theta = -S}$$

To show: $A = (I + S)(I - S)^{-1}$ is unitary matrix

$$\text{i.e. } A^\theta A = I \quad \text{or} \quad AA^\theta = I.$$

$$\begin{aligned} \text{Consider, } A^\theta &= \left((I + S)(I - S)^{-1} \right)^\theta \\ &= \left((I - S)^{-1} \right)^\theta (I + S)^\theta \\ &= \left((I - S)^\theta \right)^{-1} (I + S^\theta) \quad \left(\begin{array}{l} A^\theta = (P^{-1})^\theta \\ T^\theta = P^{-1} \\ (T^\theta)^\theta = P \end{array} \right) \\ &= (I - S^\theta)^{-1} (I - S) \end{aligned}$$

$$\text{Consider, } A^\theta A = (I + S)^{-1} (I - S) (I + S)(I - S)^{-1}$$

$$\begin{aligned} \text{Consider, } (I - S)(I + S) &= I + S - S - S^2 \\ &= I - S + S - S^2 \\ &= (I + S)(I - S) \end{aligned}$$

$$\begin{aligned} \Rightarrow A^\theta A &= (I + S)^{-1} (I + S) (I - S) (I - S)^{-1} \\ &= I \cdot I \quad (I \text{ is identity}) \end{aligned}$$

$$\boxed{A^\theta A = I} \Rightarrow A \text{ is unitary Matrix}$$

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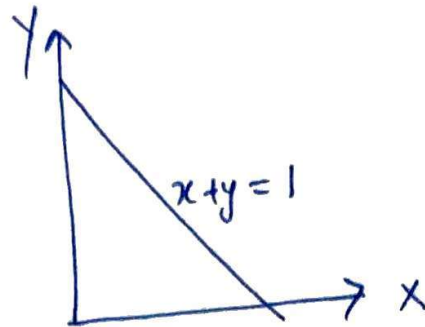
Q.7 (b)

Using the transformation $x + y = u, y = uv$, calculate $\iint \left[xy(1-x-y) \right]^{\frac{1}{2}} dx dy$, integration being taken over by the area of the triangle bounded by the lines $x = 0, y = 0, x + y = 1$. (15)

To find: $I = \iint (xy)^{\frac{1}{2}} (1-x-y)^{\frac{1}{2}} dx dy$

On region:

$$x=0, y=0 \\ x+y=1.$$



Consider transformation

$$x+y=u \quad y=uv$$

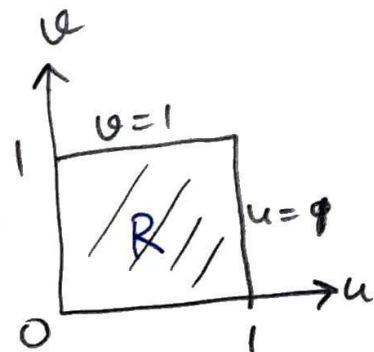
$$\Rightarrow x = u - uv = u(1-v)$$

$$x=0 \Rightarrow u=0, v=1$$

$$y=0 \Rightarrow u=0, v=0$$

$$x+y=1 \Rightarrow u=1$$

$\therefore$ In New System, Region is $\rightarrow$



and In new System area I

$$dx dy = \frac{J(x,y)}{J(u,v)} du dv$$

$$\frac{J(x,y)}{J(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u - uv + uv = u$$

$$\Rightarrow dx dy = u du dv$$

$\therefore$ I can be written as

$$I = \iint_R \left(u(1-u) uv \right)^{1/2} (1-u)^{1/2} u \, du \, dv$$

$$= \iint_R u^3 \sqrt{1-u} \, du \times v(1-v) \, dv$$

$$= \iint_R \left(u^2 v(1-v) \right)^{1/2} (1-u)^{1/2} u \, du \, dv$$

$$= \iint_R u^2 \sqrt{1-u} \, du \cdot \sqrt{v} \sqrt{1-v} \, dv$$

$$= \int_{u=0}^1 \int_{v=0}^1 u^2 \sqrt{1-u} \, du \cdot \sqrt{v} \sqrt{1-v} \, dv$$

$$= \int_0^1 u^2 \sqrt{1-u} \, du \times \int_0^1 v^{1/2} (1-v)^{1/2} \, dv$$

$$= B(3, 3/2) \times B(3/2, 3/2)$$

$$= \frac{\cancel{\Gamma(3)} \Gamma(3/2)}{\Gamma(3+3/2)} \times \frac{\Gamma(3/2) \Gamma(3/2)}{\cancel{\Gamma(3)}}$$

$$= \frac{(\Gamma(3/2))^3}{(2+3/2)(1+3/2)(3/2)\Gamma(3/2)}$$

$$= \frac{(\Gamma(3/2))^2}{7/2 \cdot 5/2 \cdot 3/2} = \frac{8}{35 \cdot 3} \left[\frac{1}{2} \Gamma(1/2) \right]^2$$

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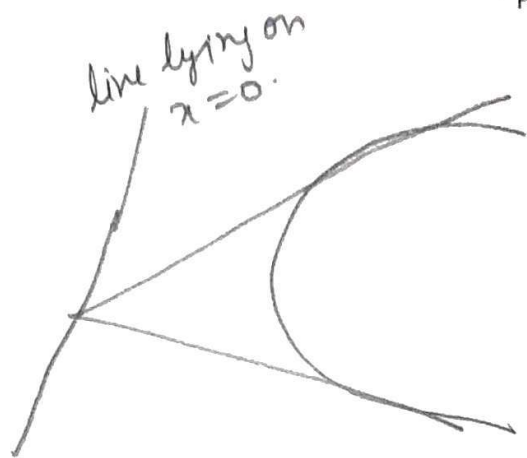
$$= \frac{8}{35 \cdot 3} \cdot \frac{1}{4} \cdot \pi = \boxed{\frac{2}{105} \pi}$$

Q.7 (c)

Two perpendicular tangent planes to the paraboloid $\frac{x^2}{a} + \frac{y^2}{b} = 2z$ intersect in a line lying on the plane $x = 0$. Prove that the line touches the parabola $x = 0, y^2 = (a+b)(2z+a)$.

let line L be lying on $x=0$ from which two $\perp$ tangent are drawn to paraboloid

$$\frac{x^2}{a} + \frac{y^2}{b} = 2z$$



Wkt, if a plane $lx + my + nz = p$ is tangent to paraboloid $ax^2 + by^2 = 2z$ then

$$\frac{l^2}{a} + \frac{m^2}{b} = -2np$$

In this case, any tangent plane to paraboloid will be

$$lx + my + nz = -\frac{1}{2n} (al^2 + bm^2)$$

there are two such plane.

$$\therefore l_1x + m_1y + n_1z = -\frac{1}{2n_1} (al_1^2 + bm_1^2)$$

$$l_2x + m_2y + n_2z = -\frac{1}{2n_2} (al_2^2 + bm_2^2)$$

there intersection lie on $x=0$.

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