

NEXT IAS

(To be filled by candidate)

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Roll No. : MT24MATL1011

Registration Number : NIAS 2400020963 Date of Examination :

Exam Centre : Old Rajinder Nagar ☒ Online ☒

CSE (MAINS) TEST SERIES - 2024

Test-01	MATHEMATICS
Dated : 23-06-2024	OPTIONAL

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

- (Please read each of the following instructions carefully before attempting questions)
- There are EIGHT questions divided in Two sections and printed in ENGLISH.
- Candidate has to attempt FIVE questions in all.
- Question No. 1 and 5 are compulsory and out of the remaining, any THREE are to be attempted choosing at least ONE question from each section.
- The number of marks carried by a question/part is indicated against it.
- Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.
- No marks will be given for answers written in a medium other than the authorized one.
- Word limit in questions, wherever specified, should be adhered to.
- Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

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(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	20			
2. (b)	18	15			
2. (c)	21	15			
3. (a)	24	15			
3. (b)	27	15			
3. (c)	30	20			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	40	20			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	15			
6. (b)	57	20			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	15			
7. (c)	70	20			
8. (a)	74	15			
8. (b)	77	20			
8. (c)	81	15			
Grand Total					

Remarks

Observations:

Consider a function $f: V \rightarrow V$ such that

$f(x) = x$ for all $x \in V$ and $f(x) = 0$ for $x \in W$.

$$V = W \oplus W^\perp$$

Let $x \in W$ and $y \in W^\perp$. Then

$$f(x+y) = f(x) + f(y) = x + 0 = x$$

$$f(x+y) = x = (x+y) - y$$

$$f(x+y) - (x+y) = -y$$

$$0 = -y \text{ for all } y \in W^\perp$$

$$0 = f(y) - y$$

$$(f(y) - y) = 0 \Rightarrow f(y) = y$$

$$f(y) = y \text{ for all } y \in W^\perp$$

Thus f is the identity map on $W^\perp$.

Conversely, let $x \in W$.

$$f(x) = 0$$

$$f(x) = 0 \text{ for all } x \in W$$

SECTION-A

Q.1 (a) If V is finite dimensional and f is a homomorphism of V onto V , prove that f must be one one and so, an isomorphism. (10)

Given $\rightarrow V$ is finite dimensional vector space and f is a homomorphism
 $f: V \rightarrow V$.

To prove: f is one one

Proof $\rightarrow$ let $x_1, x_2 \in V$

where $f(x_1) = y_1 \in V$

$f(x_2) = y_2 \in V$

to prove $x_1 = x_2$

$$f(x_1) - f(x_2) = 0$$

$$f(x_1 - x_2) = 0$$

$\Rightarrow$

$$\left(\begin{aligned} &\because f(ax + by) \\ &= af(x) + bf(y) \\ &\text{as } f \text{ is homo} \end{aligned} \right)$$

$$\text{Rank } f + \text{Nullity } f = \dim(V) \quad (\text{Rank nullity theorem})$$

$\therefore f$ is onto

$$\therefore \text{Rank } f = \dim(V)$$

Substituting

we get Nullity $f = 0$

$$\Rightarrow \text{Ker } f = \{0\}$$

$$\Rightarrow f(x_1 - x_2) = 0 \Rightarrow x_1 - x_2 \in \text{Ker } f$$

$$\therefore x_1 - x_2 = 0 \in V$$

$$x_1 = x_2$$

Hence f is one one and so,
an isomorphism.

Q.1 (b)

Determine the set of points where the function $f(x) = \frac{x}{1+|x|}$ is differentiable. (10)

$$f(x) = \begin{cases} \frac{x}{1+x} & x \geq 0 \\ \frac{x}{1-x} & x < 0, \text{ and } \end{cases}$$

Case I $x \geq 0$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

left hand derivative

$$\lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0^+} \frac{f(x-h) - f(x)}{-h}$$

$$= \lim_{h \rightarrow 0^+} \frac{\frac{x-h}{1+x-h} - \frac{x}{1+x}}{-h}$$

$$= \lim_{h \rightarrow 0^+} \frac{-h}{-h(1+x-h)(1+x)} = \frac{1}{(1+x)^2}$$

Right hand derivative

$$\lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{\frac{x+h}{1+x+h} - \frac{x}{1+x}}{h}$$

$$= \frac{x(1+x) + h(1+x) - x(1+x+h)}{h(1+x+h)(1+x)}$$

$$= \lim_{h \rightarrow 0^+} \frac{1}{(1+x+h)(1+x)} = \frac{1}{(1+x)^2}$$

LHD = RHD

Case II. $x = 0$

$$\text{LHD} = \lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0^+} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \rightarrow 0^+} \frac{-h}{(1+h)(-h)} = 1$$

$$\text{RHD} = \lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h}{1+h} = 1$$

Case III. $x < 0$

$$\text{LHD} = \lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0^+} \frac{x-h}{1-x+h} = \frac{x}{1-x}$$

$$= \lim_{h \rightarrow 0^+} \frac{-h}{(1-x+h)(1-x)(-h)} = \frac{1}{(1-x)^2}$$

$$\text{RHD} = \lim_{h \rightarrow 0^+} \frac{\frac{x+h}{1-x-h} - \frac{x}{1-x}}{h} = \frac{1}{(1-x)^2}$$

Hence differentiable for all x
i.e. $x \in \mathbb{R}$

Q.1 (c)

Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \cot^2 x \right)$

(10)

$$L = \lim_{x \rightarrow 0} \frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x}$$

$\left(\frac{0}{0} \text{ form} \right)$

Using L'Hopital

i.e.

$$\lim_{x \rightarrow 0} \frac{f}{g} = \lim_{x \rightarrow 0} \frac{f'}{g'} \quad \text{when } \frac{f(x)}{g(x)} \text{ is } \frac{0}{0} \text{ form}$$

$$L = \lim_{x \rightarrow 0} \frac{\sin 2x - 2x \cos^2 x + x^2 \sin 2x}{2x \sin^2 x + x^2 \sin 2x}$$

Again using L'Hopital

$$L = \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2 \cos^2 x + 2x \sin 2x + 2x \sin 2x + x^2 (2 \cos 2x)}{2 \sin^2 x + 2x \sin 2x + 2x \sin 2x + x^2 (2 \cos 2x)}$$

$$= \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2 \sin^2 x + 4x \sin 2x + x^2 (2 \cos 2x)}{2 \sin^2 x + 4x \sin 2x + x^2 (2 \cos 2x)}$$

Dividing Numerator and denominator by x^2 and using $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Candidates must not write on this margin

We get

At $x \rightarrow 0$

$$\frac{-2\left(\frac{\sin x}{x}\right)^2 + 8\left(\frac{\sin 2x}{2x}\right) + 2\cos 2x}{2\left(\frac{\sin x}{x}\right)^2 + 8\frac{\sin 2x}{2x} + 2\cos 2x}$$

$$= \frac{-2 + 8 + 2}{2 + 8 + 2} = \frac{8}{12} = \frac{2}{3}$$



Q.1 (d)

Find the centre of the conic $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1, 2x + 2y - z = 3$.

(10)

let x_1, y_1, z_1 be the centre of the given conic.

We know equation of section is given by $T = S_1$ where T is tangent plane at x_1, y_1, z_1 and S_1 is $\frac{x_1^2}{9} + \frac{y_1^2}{16} + \frac{z_1^2}{4} - 1$

$$T = \frac{xx_1}{9} + \frac{yy_1}{16} + \frac{zz_1}{4} - 1 = S_1$$

We get

$$\frac{xx_1}{9} + \frac{yy_1}{16} + \frac{zz_1}{4} - 1 - \left(\frac{x_1^2}{9} + \frac{y_1^2}{16} + \frac{z_1^2}{4} - 1 \right) = 0$$

As x_1, y_1, z_1 are centre of conic

As given plane $2x + 2y - z = 3$ and (i) are identical

Comparing we get

Candidates must not write on this margin

$$\frac{2}{\frac{x_1}{9}} = \frac{2}{y_1}$$

$$\frac{x_1}{9(2)} = \frac{y_1}{16(2)} = \frac{z_1}{4(-1)} = \frac{\frac{x_1^2}{9} + \frac{y_1^2}{16} + \frac{z_1^2}{4}}{3}$$

Now,

$$\frac{x_1}{9} = \frac{y_1}{16} \Rightarrow x_1 = \frac{9y_1}{16}$$

$$\frac{z_1}{(-4)} = \frac{y_1}{16(2)} \Rightarrow z_1 = -\frac{y_1}{8}$$

$$\Rightarrow \frac{y_1}{16(2)} = \frac{\frac{9y_1^2}{16^2} + \frac{y_1^2}{16} + \frac{y_1^2}{64 \times 4}}{3}$$

$$\frac{3y_1}{2} = \frac{9y_1^2}{16} + 1 + \frac{1}{16} = y_1^2 \left[\frac{13}{8} \right]$$

either $y_1 = 0$ or $y_1 = \frac{12}{13}$

$\Rightarrow$ ~~but~~ $x_1 = 0$ or $x_1 = \frac{27}{52}$

$z_1 = 0$ or $z_1 = -\frac{3}{26}$

$(0, 0, 0)$ not satisfy given plane

$\therefore$ centre is $\left(\frac{27}{52}, \frac{12}{13}, -\frac{3}{26} \right)$

Q.1 (e)

Find the equation of the cylinder whose generators touch the sphere $x^2 + y^2 + z^2 = 9$ and are perpendicular to the plane $x - y - 3z = 5$. (10)

Dir's of axis are $1, -1, -3$

$\Rightarrow$ any generator is

$$\frac{x - x_1}{1} = \frac{y - y_1}{-1} = \frac{z - z_1}{-3} = r$$

$$x = (x, y, z) = (x_1 + r, y_1 - r, z_1 - 3r)$$

touches the sphere at above point

$$(x_1 + r)^2 + (y_1 - r)^2 + (z_1 - 3r)^2 = 9$$

$$11r^2 + r(2x_1 - 2y_1 - 6z_1) + x_1^2 + y_1^2 + z_1^2 - 9 = 0$$

$\therefore$ it touches the two values of r should be identical

$$\therefore D = B^2 - 4AC = 0$$

$$(2x_1 - 2y_1 - 6z_1)^2 - 4(x_1^2 + y_1^2 + z_1^2 - 9) = 0$$

$$(x_1 - y_1 - 3z_1)^2 = 4(x_1^2 + y_1^2 + z_1^2 - 9)$$

$$10(x_1^2 + y_1^2) + 2z_1^2 = 99 - 2x_1y_1 + 6y_1z_1 - 6x_1z_1$$

Required Equation is

$$10(x^2 + y^2) + 2z^2 - 99 + 2xy - 6yz + 6xz = 0$$

Q.2 (a)

Let V be vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let V_e be the set of all even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions, $f(-x) = -f(x)$.

i. Prove that V_e and V_o are subspaces of V .

ii. Prove that $V_e + V_o = V$

iii. Prove that $V_e \cap V_o = \{0\}$.

(20)

$\Rightarrow$ To prove V_e is subspace of V

let $m, n \in V_e$

Consider, $\alpha m + \beta n$

where $\alpha, \beta \in \mathbb{R}$

$$\cancel{(\alpha m + \beta n)(x)} = \cancel{(\alpha m)(x)} + \cancel{(\beta n)(x)}$$

~~Using the fact that~~

$\alpha m \in V$ as $\alpha \in \mathbb{R}$ and $m \in V_e \subset V$

$\beta n \in V$ as $\beta \in \mathbb{R}$ and $n \in V_e \subset V$

$$(\alpha m + \beta n)(x) = (\alpha m)(x) + (\beta n)(x)$$

as $\alpha m \in V, \beta n \in V$

$\therefore$ using the property of vector space that $(v_1 + v_2)(x) = v_1(x) + v_2(x)$

$$= \alpha(m(-x)) + \beta(n(-x))$$

~~as~~ using property of functions

$$= \alpha(m(x)) + \beta(n(x))$$

Using property of even functions

$$= (\alpha m)(x) + (\beta n)(x) = (\alpha m + \beta n)(x)$$

$\Rightarrow V_e$ is subspace.

To prove V_0 is subspace of V

Consider $r, s \in \mathbb{R}; p, q \in V_0$

$$\begin{aligned} (rp + sq)(-x) &= rp(-x) + sq(-x) \\ &= r[p(-x)] + s[q(-x)] \\ &= r[-p(x)] + s[-q(x)] \\ &\quad \text{using property of odd functions} \\ &= -(rp(x)) - (sq(x)) \\ &= -[rp(x) + sq(x)] \\ &= -(rp + sq)(x) \end{aligned}$$

$\Rightarrow rp + sq \in V_0 \Rightarrow V_0$ is subspace

$$ii) V_e + V_0 = \{v_e + v_0 \mid v_e \in V_e, v_0 \in V_0\}$$

$$v_e \in V_e \subset V$$

$$v_0 \in V_0 \subset V$$

$$\Rightarrow v_e, v_0 \in V$$

$$\Rightarrow v_e + v_0 \in V \Rightarrow V_e + V_0 \subset V$$

~~let $v \in V$~~

$$\dim(V_e + V_o) = \dim(V_e) + \dim(V_o) - \dim(V_e \cap V_o)$$

V_e is a subspace of V and V_o is a subspace of V .

Consider $V_e \cap V_o$ is a subspace of V .

$$(x-y)p_2 + (x-y)q_1 = (x-y)(p_2 + q_1)$$

$$[(x-y)p_2] + [(x-y)q_1] =$$

$$[(x-y)p_2] + [(x-y)q_1] =$$

$$(x)(p_2) + (x)(q_1) =$$

$$[(x)p_2] + [(x)q_1] =$$

$$[(x)] [p_2 + q_1] =$$

$$\text{suppose } x \in V_e \text{ and } p_2 + q_1 \in V_o$$

$$\left(\begin{matrix} x \in V_e \\ p_2 + q_1 \in V_o \end{matrix} \right) \Rightarrow x(p_2 + q_1) \in V_e + V_o$$

$$V_e + V_o \subseteq V$$

$$V_e + V_o \subseteq V$$

$$V_e + V_o \subseteq V$$

$$\text{iii)} \quad V_e \cap V_o = \{0\}$$

$$\text{let } f \in V_e \cap V_o$$

$$\Rightarrow f \in V_e \quad \text{and} \quad f \in V_o$$

$$\Rightarrow f(x) = f(-x) \quad \text{and} \quad f(-x) = -f(x) \quad \forall x$$

$$\Rightarrow f(x) = -f(x) \quad \forall x$$

$$\Rightarrow 2f(x) = 0 \quad \forall x$$

$$\Rightarrow f(x) = 0 \quad \forall x$$

$$\Rightarrow V_e \cap V_o = \{0\}.$$

Q.2 (b) Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$ for $0 < x < \frac{\pi}{2}$.

(15)

Consider

$$f = \frac{\tan x}{x} - \frac{x}{\sin x}$$

$$f = \tan x \sin x - x^2$$

$$f'(x) = \sec^2 x \sin x + \sin x - 2x$$

$$\begin{aligned} f''(x) &= 2 \sec x (\sec x \tan x) \sin x \\ &\quad + \sec^2 x \cos x + \cos x - 2 \\ &= 2 \sec x \tan^2 x + \sec x + \cos x - 2 \\ &= 2 \frac{\sin^2 x}{\cos^3 x} - 1 + \sec x - 1 + \cos x \\ &= \frac{2 \sin^2 x - \cos^3 x}{\cos^3 x} \end{aligned}$$

- (c) Find the equation of two straight lines through the point $(1, 1, 1)$ that intersect the line $x-4 = 2(y-4) = 2(z-1)$ at an angle of 60° . (15)

let the line through $(1, 1, 1)$ be

$$\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-1}{n}$$

~~Any point is~~
 ~~(x, y, z)~~

This intersects the
given line

$\therefore$ condition of coplanarity

$$\begin{vmatrix} 3 & 3 & 0 \\ l & m & n \\ 1 & 1/2 & 1/2 \end{vmatrix} = 0$$

$$\left[\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} ; \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \right]$$

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \frac{3}{2}(m-n) - 3\left(\frac{l}{2} - n\right) = 0$$

$$(m-n) - (l-2n) = 0$$

$$m - l + n = 0 \quad -i$$

The lines intersect at 60°

$$\therefore \cos 60 = \frac{l + \frac{m}{2} + \frac{n}{2}}{\sqrt{l^2 + \frac{1}{2^2} + \frac{1}{2^2}} \sqrt{l^2 + m^2 + n^2}}$$

$$\left[\cos \theta = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \sqrt{l_2^2 + m_2^2 + n_2^2}} \right]$$

$$\therefore l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \begin{cases} 2l + m + n = \frac{\sqrt{3}}{2} & \text{--- i} \\ -l + m + n = 0 & \text{--- ii} \end{cases}$$

Solving

$$l = m + n$$

$$3(m + n) = \frac{\sqrt{3}}{\sqrt{2}}$$

$$m + n = \frac{1}{\sqrt{6}} = l$$

$$l^2 + m^2 + n^2 = 1$$

$$\left(\frac{1}{\sqrt{6}}\right)^2 + m^2 + \left(\frac{1}{\sqrt{6}} - m\right)^2 = 1$$

$$\frac{1}{3} + 2m^2 - \frac{\sqrt{2}}{3} m = 1$$

$$m^2 - \frac{m}{\sqrt{6}} - \frac{1}{3} = 0$$

$$m = \frac{\frac{1}{\sqrt{6}} + \sqrt{\frac{1}{6} + \frac{4}{3}}}{2}$$

$$= \frac{\frac{1}{\sqrt{6}} + \frac{\sqrt{10}}{\sqrt{6}}}{2} = \frac{1 + \sqrt{10}}{2\sqrt{6}}$$

$$n = \frac{1}{\sqrt{6}} - m = -\frac{1}{\sqrt{6}}, \frac{\sqrt{10}}{\sqrt{6}}$$

The two eqⁿ are

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{-1}$$

$$\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z-1}{2}$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 1(2-1) = 1$$

$$0 + 8 - (1 + 2) = 5$$

$$(0, 0, 0) + (x, y, z) = (x, y, z)$$

Point of intersection of the lines is

$$(1, 2, -1)$$

Q.3 (a) Let T be a linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$

Prove that T is invertible and find the formula for T^{-1} .

(15)

1 let $(x_1, x_2, x_3) \in \text{Ker } T$
 $T(x_1, x_2, x_3) = (0, 0, 0)$

$$\begin{aligned} 3x_1 + 0x_2 + x_3 &= 0 \\ -2x_1 + x_2 + 0x_3 &= 0 \\ -x_1 + 2x_2 + 4x_3 &= 0 \end{aligned}$$

$$AX = 0$$

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \quad X \neq 0$$

$$|A| = \begin{vmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{vmatrix}$$

$$= 3(4) + 1(-4 + 1) = 8 \neq 0$$

$$\Rightarrow (x_1, x_2, x_3) = (0, 0, 0)$$

is only solution for $AX = 0$

$$\therefore \text{Ker } T = \{0\}$$

$\Rightarrow T$ is one one

as let $x_1, x_2 \in \mathbb{R}^3$ such that

$$T(x_1) = T(x_2) \Rightarrow T(x_1) - T(x_2) = 0$$

$$\Rightarrow T(x_1 - x_2) = 0 \quad (\text{by property of linear transformation})$$

$$\Rightarrow x_1 - x_2 \in \text{Ker } T$$

$$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

i.e. T is one one.

Also $\rho(T) + \nu(T) = \dim(\mathbb{R}^3)$
(Rank Nullity theorem)
 $\Rightarrow \rho(T) = \dim(\mathbb{R}^3)$

$\Rightarrow T$ is onto

$\therefore T$ is one one and onto

$\therefore T$ is invertible.

To find T^{-1}

$$3x_1 + x_3 = a \quad \text{--- i}$$

$$2x_1 + x_2 = b \quad \text{--- ii}$$

$$x_1 + 2x_2 + 4x_3 = c \quad \text{--- iii}$$

from i, ii

$$x_3 = a - 3x_1; \quad x_2 = b + 2x_1$$

put in 'iii'

$$x_1(-1 + 2b + 4a - 12) = c - 4a - 2b$$

$$x_1 = \frac{4a + 2b - c}{9}$$

$$x_2 = b + 2x_1 = \frac{8a + 13b - 2c}{9}$$

$$x_3 = a - 3x_1 = \frac{5a - 6b + 3c}{9}$$

$$\therefore T^{-1}(a, b, c) = \left(\frac{4a + 2b - c}{9}, \frac{8a + 13b - 2c}{9}, \frac{5a - 6b + 3c}{9} \right)$$

- (b) Let $g(x) = f(x) + f(1-x)$ and $f''(x) > 0, \forall x \in (0,1)$. Find the interval of increase and decrease of $g(x)$. (15)

$$g'(x) = f'(x) - f'(1-x)$$

$$g''(x) = f''(x) + f''(1-x)$$

~~$$g'(x) = 0 \Rightarrow f'(x) = f'(1-x)$$~~

$\Rightarrow$

$$g''(x) > 0 \quad \text{as } f''(x) > 0$$

$$\text{and } f''(1-x) > 0 \text{ as}$$

$$1-x \in (0,1)$$

$$\text{as } x \in (0,1)$$

$$\Rightarrow g'(x) > g'(0)$$

Also $f'(x) > f'(y) \quad \forall x > y$
 $x, y \in (0,1)$
 i.e. $f'(x) > f'(\frac{1}{2}) \quad \forall x \in (\frac{1}{2}, 1)$

$$\text{and } f'(x) < f'(\frac{1}{2}) \quad \forall x \in (0, \frac{1}{2})$$

$$\Rightarrow g'(x) > 0 \quad \forall x \in (\frac{1}{2}, 1)$$

$$g'(x) < 0 \quad \forall x \in (0, \frac{1}{2})$$

$$g'(x) = 0 \quad x = \frac{1}{2}$$

$\therefore x \in (0, \frac{1}{2})$ is interval of decrease and $x \in (\frac{1}{2}, 1)$ is of increase

Find the equation of the cone with vertex at $(2a, b, c)$ and passing through the curve $x^2 + y^2 = 4a^2$ and $z = 0$. Find b and c if the cone also passes through the curve $y^2 - 4a(z+a), x = 0$. Also show that the cone is cut by the plane $y = 0$ in two straight lines and the angle θ between them is given by $\tan \theta = 2$. (20)

let eqⁿ of generator of cone with ~~the~~ direction cosine l, m, n

$$\text{be } \frac{x-2a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \quad \dots$$

Meets the curve at

$$\frac{x-2a}{l} = \frac{y-b}{m} = \frac{-c}{n}$$

$$(x, y) \Rightarrow \left(2a - \frac{lc}{n}, b - \frac{mc}{n} \right)$$

$$x^2 + y^2 = 4a^2$$

$$\left(2a - \frac{lc}{n} \right)^2 + \left(b - \frac{mc}{n} \right)^2 = 4a^2$$

$$\frac{c^2}{n^2} (l^2 + m^2) - \frac{4alc}{n} - \frac{4bcm}{n} = 0$$

Using -i

$$c^2 \left((x-2a)^2 + (y-b)^2 \right) - 4ac(x-2a)(z-c)$$

$$- 4bc(y-b)(z-c) = 0$$

is equation of cone.

If ~~cone~~ cone pass through
 ① $x = 0$ and $y^2 = 4a(z + a)$ then

putting $x = 0$

$$c^2(4a^2 + (y-b)^2) + 2a^2c(z-c)$$

$$- 4bc(y-b)(z-c) = 0$$

$$(y-b)^2 c^2 = \frac{1}{c^2} \{ 4bc(y-b)(z-c) - 2a^2c(z-c) - 4a^2c^2 \}$$

comparing with $y^2 = 4a(z + a)$

Coefficient of $y^2 = 1$; $z = 4a$ and,
 $yz = 0$, $y = 0$, constant term is $4a^2$

$$4bc = 0; \quad 4bc = 0; \quad -\frac{2a^2c}{c} = 4a;$$

$$\Rightarrow b = 0; \quad c = -2a$$

putting $y = 0$ in equation of cone

$$(4a^2) [x-2a]^2 + 2a^2(x-2a)(z+2a) = 0$$

$$(x-2a) [x(x-2a) + 2(z+2a)] = 0$$

$$(x-2a)(x^2 + 2z + 2a) = 0$$

$$x = 2a; \quad z = -2x + 2$$

$$x = -\frac{2z}{2} + 2$$

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{0 - (-2)}{1 + (0)(-2)} = 2$$

SECTION-B

- Q.5 (a) The eigenvalues of a real symmetric matrix A are 1, 1 and 2. The corresponding eigenvectors are $\frac{1}{\sqrt{2}}(-1 \ 1 \ 0)^T$, $(0 \ 0 \ 1)^T$ and $\frac{1}{\sqrt{2}}(-1 \ -1 \ 0)^T$ respectively. Find the matrix A^4 . (10)

Characteristic equation is

$$(\lambda - 1)(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

Using Cayley Hamilton theorem

$$A^3 - 4A^2 + 5A - 2I = 0$$

$$\Rightarrow A^4 = 4A^3 - 5A^2 + 2A$$

$$= 4[4A^2 - 5A + 2I] - 5A^2 + 2A$$

$$A^4 = 11A^2 - 18A + 8I$$

Also

$$Ax = \lambda x$$

$$A \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \text{ for } \lambda = 1$$

$$A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ for } \lambda = 1$$

$$A \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = 2 \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \text{ for } \lambda = 2$$

$$\text{let } A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & b \end{bmatrix}$$

Using $\lambda = 1$

$$-a + b = -1$$

$$-b + d = 1$$

$$-c + e = 0$$

Using $\lambda = 1$

$$c = e = 0 \text{ if } b = 1$$

Using $\lambda = 2$

$$-a - b = 2(-1)$$

$$-b - d = 2(-1)$$

$$-c - e = 2(0)$$

$$-2a = -3 \Rightarrow a = 3/2 \Rightarrow b = 1/2$$

$$-2b = -1 \Rightarrow b = 1/2 \Rightarrow d = 3/2$$

$$\Rightarrow A = \begin{bmatrix} 3/2 & 1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 5/2 & 3/2 & 0 \\ 3/2 & 5/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^4 = 11A^2 - 18A + 8I$$

$$= \begin{bmatrix} 17/2 & 15/2 & 0 \\ 15/2 & 17/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q.5 (b) Show that $\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} \log(1+a), a \geq 0$

$$I = \int_0^a \frac{\tan^{-1} ax}{x(1+x^2)} dx + \int_a^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx$$

$I_1 + I_2$

$$\lim_{x \rightarrow 0^+} \frac{\tan^{-1} ax}{x(1+x^2)} = a \left(\text{as } \lim_{x \rightarrow 0^+} \frac{\tan^{-1} x}{x} = 1 \right)$$

I_1 is convergent

I_2 let $f = \frac{\tan^{-1} ax}{x(1+x^2)}$

$g = \frac{1}{x^{1+1/2}}$

$$\lim_{x \rightarrow \infty} \frac{f}{g} = \lim_{x \rightarrow \infty} \frac{x^{1/2} \tan^{-1} ax}{x(1+x^2)}$$

$$\lim_{x \rightarrow \infty} \frac{\tan^{-1} ax}{x^{-1/2} + x^{3/2}} = \frac{\pi}{2(0+\infty)} = 0$$

$\Rightarrow f, g$ converge or diverge together

$\int_a^{\infty} g dx$ converges as $p > 1$

$\therefore \int_a^{\infty} f dx$ converges

Now $I = \int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx$

let $ax = \tan \theta$
 $adx = \sec^2 \theta d\theta$

$$I = \int_0^{\pi/2} \frac{\theta \sec^2 \theta}{a \left(\frac{\tan \theta}{a} \right) \left(\frac{a^2 + \tan^2 \theta}{a^2} \right)} d\theta$$

$$= \int_0^{\pi/2} \frac{a^2 \theta \sec^2 \theta d\theta}{(a^2 + \tan^2 \theta)(\tan \theta)}$$

$$= a^2 \left[\theta \int_0^{\pi/2} \frac{\sec^2 \theta d\theta}{(a^2 + \tan^2 \theta)(\tan \theta)} - \int_0^{\pi/2} \frac{\sec^2 \theta d\theta}{\tan \theta (a^2 + \tan^2 \theta)} \right]$$

let $\tan \theta = t$
 $\sec^2 \theta d\theta = dt$

$$\int \frac{\sec^2 \theta d\theta}{(a^2 + \tan^2 \theta)(\tan \theta)} = \int \frac{dt}{t(a^2 + t^2)}$$

Q.5 (c)

Prove that the function $f(x,y) = \sqrt{|xy|}$ is not differentiable at the point $(0,0)$, but f_x and f_y both exist at the origin and have the value 0. (10)

$$f(0+h, 0+k) - f(0,0) = hA + kB + \sqrt{h^2+k^2} \phi(h,k)$$

where $\text{let } h, k \rightarrow (0,0)$

and $\phi(h,k) \rightarrow (0,0)$

A, B independent of h, k

$$\sqrt{|hk|} = Ah + Bk + \sqrt{h^2+k^2} \phi(h,k)$$

~~$\phi(h,k)$~~ let $k=0$ and h approach from $+x$ axis

$$0 = Ah + \cancel{Bk} + \sqrt{h^2+0} \phi(h,0)$$

$$\Rightarrow A = -\phi(h,0) = -\phi(0,0)$$

let $h=0$ and k approach from $+y$ axis

$$\Rightarrow B = -\phi(0,0) = 0$$

let $h=k$ and both approach 0 from positive side

$$\Rightarrow h = -\phi(0,0)h + \sqrt{2}h \phi(h,k)$$

$$h(1 + \phi(0,0) - \sqrt{2} \phi(h,k)) = 0$$

$$\Rightarrow \sqrt{2} \phi(h,k) + 1 = 0$$

$$\Rightarrow \phi(h,k) = -1/\sqrt{2}$$

$$\text{as } (h, k) \rightarrow (0, 0)$$

$$\text{i.e. } \phi(h, k) \neq 0$$

$\rightarrow f(x, y)$ is not differentiable

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$$

$$f_y(0, 0) = \lim_{k \rightarrow 0} \frac{f(0, 0+k) - f(0, 0)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{0 - 0}{k} = 0$$

Q.5 (d)

Prove that the equations of the chord through the point $(1, 2, 3)$ which is bisected by the diametral plane $10x - 24y = 21$ of the paraboloid $5x^2 - 6y^2 = 7z$ are

$$(x-1) - \frac{1}{2}(y-2) - \frac{1}{3}(z-3).$$

(10)

let equation of chord be

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n}$$

$$SS_1 = T^2$$

$$(5x^2 - 6y^2 - 7z)(5 - 24 - 21) = \left[5x - 12y - \frac{7}{2}(3+3) \right]^2$$

$$-42(5x^2 - 6y^2 - 7z) = 5x - 12y - \frac{7}{2}(3+3)$$

Q.5 (e)

Find the equation of the tangent planes to the hyperboloid $2x^2 - 6y^2 + 3z^2 = 5$ which pass through the line $x + 9y - 3z = 0 = 3x - 3y + 6z - 5$. (10)

Any plane through the two lines is
 $L_1 + \lambda L_2$

$$x + 9y - 3z + \lambda(3x - 3y + 6z - 5) = 0$$

$$x(1 + 3\lambda) + y(9 - 3\lambda) + z(-3 + 6\lambda) - 5\lambda = 0$$

It is tangent to hyperboloid

$$\therefore \frac{(1+3\lambda)^2}{2} + \frac{(9-3\lambda)^2}{1-6} + \frac{(6\lambda-3)^2}{3} = \frac{(5\lambda)^2}{5}$$

$$\lambda^2 \left[\frac{9}{2} - \frac{9}{6} + \frac{36}{3} - 5 \right] + \lambda [3 + 9 - 12] = 0$$

$$\lambda \left(\frac{1}{2} - \frac{27}{2} + 3 \right) = 0$$

$$\lambda^2 [10] - 10 = 0$$

$$\lambda = \pm 1$$

$\therefore$ Tangent planes are

$$x(4) + y(6) + z(3) - 5 = 0$$

and

$$x(-2) + y(12) + z(-9) + 5 = 0$$

Q.8 (a)

Let V be complex vector space of 3×3 skew symmetric matrices with complex entries i.e.

$V = \{A \in M_{3 \times 3}(\mathbb{C}) \mid A^t = -A\}$. Let $B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Define a linear transformation $T: V \rightarrow V$ by $T(A) = BA - AB$. Find the eigen values and eigenvectors of T . (15)

$$A = \begin{bmatrix} 0 & b_1 + b_2 i & c_1 + c_2 i \\ -b_1 - b_2 i & 0 & d_1 + d_2 i \\ -c_1 - c_2 i & -d_1 - d_2 i & 0 \end{bmatrix}$$

$$A = -A^T$$

Basis of V : $\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$

$e_1 \quad e_2 \quad e_3$

$$T(e_1) = 0$$

$$T(e_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = -e_3$$

$$T(e_3) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = e_2$$

Matrix of linear transformation is

$$[T]_{\text{Basis of } V} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$[T - \lambda I] = 0$$

$$\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 + 1) = 0$$

$$\Rightarrow \lambda = 0, \pm i$$

Eigen vector for $\lambda = 0$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow x_2 = x_3 = 0, \quad x_1 = k$$

i.e. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

for $\lambda = i$

$$\begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0; \quad -ix_2 + x_3 = 0$$

$$-x_2 - ix_3 = 0$$

$$x_3 = ix_2; \quad x_2 = k$$

i.e. $\begin{bmatrix} 0 \\ k \\ ik \end{bmatrix}^T$

for $\lambda = -i$

$$\begin{bmatrix} i & 0 & 0 \\ 0 & i & 1 \\ 0 & -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 = 0 \quad ; \quad ix_2 + x_3 = 0$$

$$x_2 = -ix_3$$

$$x_3 = k$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & i \end{bmatrix}^T$$

Eigen Vectors = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \right\}$

Trace the curve $y^2 x^2 = x^2 - a^2$.

(20)

Symmetric about x, y axis.Not symmetric about $x=y$ line

$$y^2 = 1 - \frac{a^2}{x^2}$$

$$y^2 \geq 0 \Rightarrow x^2 \geq a^2$$

$$|x| \geq a \quad \text{--- i}$$

$$\text{Also } x^2 = \frac{-a^2}{y^2 - 1} = \frac{a^2}{1 - y^2}$$

$$\Rightarrow 1 - y^2 > 0 \Rightarrow y^2 < 1$$

$$\Rightarrow |y| < 1 \quad \text{--- ii}$$

$$y^2 = 1 - \frac{a^2}{x^2}$$

$$2y \frac{dy}{dx} = \frac{2a^2}{x^3} \Rightarrow y' = \frac{a^2}{x^3 y}$$

$$\Rightarrow = \frac{a^2}{x^3 \sqrt{1 - \frac{a^2}{x^2}}} = \frac{|x| a^2}{x^3 \sqrt{x^2 - a^2}}$$

$$y' \neq 0 \Rightarrow x \neq \pm a$$

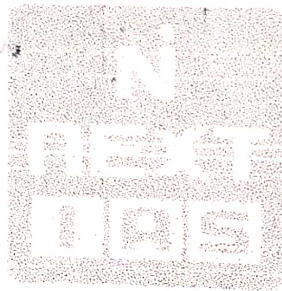
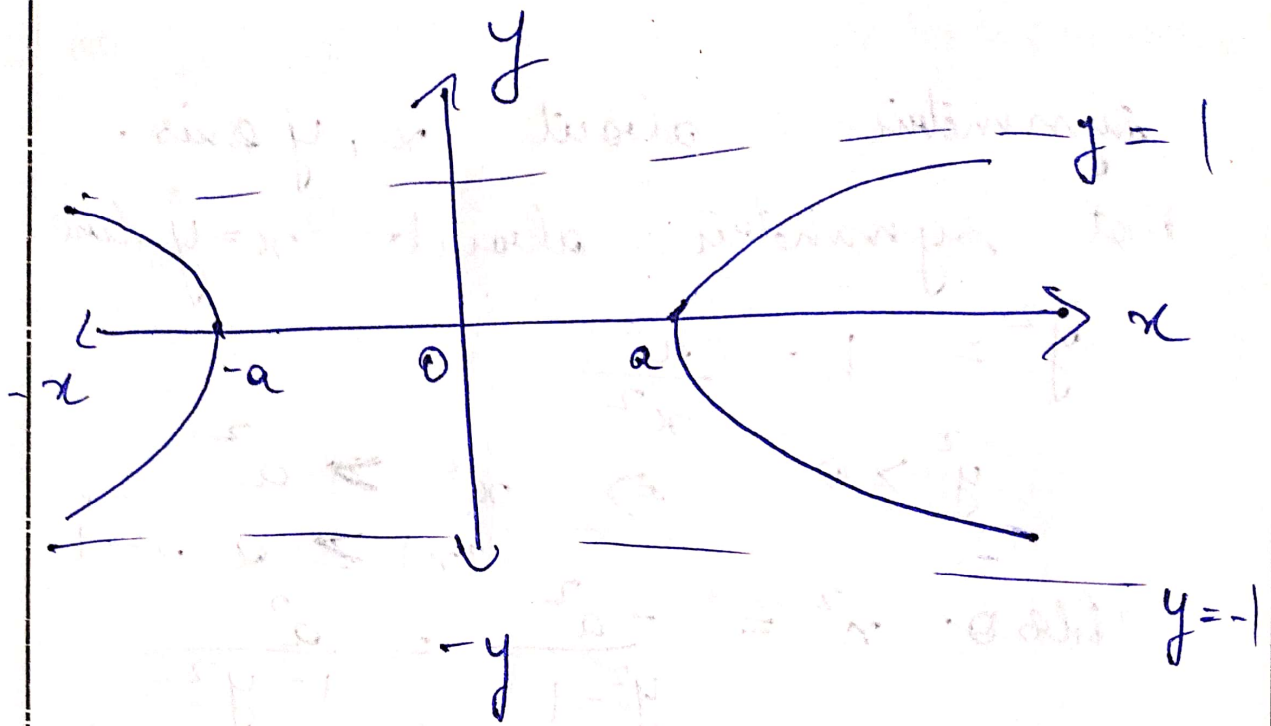
$$y' > 0 \quad \forall \quad x > a$$

$$y' < 0 \quad \forall \quad x < -a$$

Also

$$\text{As } x \rightarrow \infty ; y \rightarrow 1$$

$$\Rightarrow y = 1 \text{ is asymptote.}$$



Find the length of the normal chord through P of the ellipsoid $\sum \frac{x^2}{a^2} = 1$ and prove that if it is equal to $4PG_3$, where G_3 is the point where the normal chord through P meets the XY plane, then P lies on the cone $\frac{x^2}{a^6}(2c^2 - a^2) + \frac{y^2}{b^6}(2c^2 - b^2) + \frac{z^2}{c^4} = 0$. (15)

let $P = (x_1, y_1, z_1)$

$$\frac{x - x_1}{p \frac{x_1}{a^2}} = \frac{y - y_1}{p \frac{y_1}{b^2}} = \frac{z - z_1}{p \frac{z_1}{c^2}}$$

let $G_3 = (\cancel{0}, \cancel{0}, 0)$ $\alpha, \beta, 0$

$$PG_3 = \frac{-z_1}{p \frac{z_1}{c^2}} = -\frac{c^2}{p}$$

let length of normal chord = r

$$x = \frac{p x_1}{a^2} r + x_1; \quad y = \frac{p y_1}{b^2} r + y_1;$$

$$z = \frac{p z_1}{c^2} r + z_1$$

lie on ellipsoid.

$$\left(\frac{p x_1}{a^2} r + x_1 \right)^2 + \left(\frac{p y_1}{b^2} r + y_1 \right)^2 + \left(\frac{p z_1}{c^2} r + z_1 \right)^2 = 1$$

$$\sum \frac{x_1^2}{a^2} + \sum \frac{p^2 r^2 x_1^2}{a^6} + 2 \sum \frac{p x_1^2 r}{a^4} = 1$$

$$1 + r^2 p^2 \sum \frac{x_1^2}{a^6} + \frac{2r}{p} = 1$$

$$\Rightarrow r = \frac{-2}{p^3 \left(\sum \frac{x_i^2}{a^6} \right)}$$

now $r = 4PG_3$

$$\frac{-2}{p^3 \left(\sum \frac{x_i^2}{a^6} \right)} = 4 \left(\frac{-c^2}{p} \right)$$

$$\sum \frac{x_i^2}{a^6} = \frac{1}{2c^2 p^2}$$

$$\sum \frac{x_i^2}{a^6} = \frac{1}{2c^2} \left[\frac{4 \left(\frac{x_i^2}{a^4} \right)}{\cancel{4}} \right]$$

$$2c^2 \sum \frac{x_i^2}{a^6} = \sum \frac{x_i^2}{a^4}$$

$$2c^2 \sum \frac{x_i^2}{a^6} - \sum \frac{x_i^2}{a^4} = 0$$

is the locus of P

i.e.

$$2c^2 \sum \frac{x^2}{a^6} - \sum \frac{x^2}{a^4} = 0$$

i.e.

$$\frac{x^2}{a^6} (2c^2 - a^2) + \frac{y^2}{b^6} (2c^2 - b^2) + \frac{z^2}{c^4} = 0$$