NEXT IAS

I-II 7

(To be filled by candidate)

Name : MADHU KANT

Rol No. : MT24MATRA007 Date of Exam.: 26-Aug-2024

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Exam Centre : Old Rajinder Nagar ☑ Bhopal ☐ Jaipur ☐
Mukherjee Nagar ☐ Prayagraj ☐ Online ☐

anubhav-2024

Paper-II

Dated : 26-08-2024

MATHEMATICS OPTIONAL

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

*There are **EIGHT** questions divided in **Two sections** and printed in **ENGLISH**.*

*Candidate has to attempt **FIVE** questions in all.*

***Question No. 1 and 5** are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.*

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	15			
2. (c)	20	20			
3. (a)	24	20			
3. (b)	28	15			
3. (c)	31	15			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	40	20			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	20			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	80	20			
Grand Total					

Remarks

Observations:

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SECTION-A

1. (a) Show that a group G satisfying $a^2 = e \forall a \in G$ must be abelian. (10)

Let G be a group which satisfy

$$a^2 = e \quad \forall a \in G$$

To prove: G is abelian

let $a, b \in G$

$\therefore ab \in G$ (Closure)

$\Rightarrow (ab)^2 = e$ (given property)

& also $a^2 = e$ and $b^2 = e$, $\because a, b \in G$

$\Rightarrow (ab)(ab) = e \cdot e$

$= a^2 \cdot b^2$

$abab = aabb$

$\because$ Group follow left and right Cancellation

$\therefore abab = aabb$

$\Rightarrow \boxed{ba = ab} \quad \forall a, b \in G$

$\Rightarrow G$ is abelian

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1. (b) If the subsequences $\langle a_{2n-1}\rangle$ and $\langle a_{2n}\rangle$ of a sequence $\langle a_n\rangle$ converges to the same limit l, the sequence $\langle a_n\rangle$ converges to l. (10)

let $\langle a_n\rangle$ be a sequence s.t

subsequences $\langle a_{2n-1}\rangle$ & $\langle a_{2n}\rangle$ converges to l

To prove: $\langle a_n\rangle$ converges to l

$\therefore$ $\langle a_{2n}\rangle$ and $\langle a_{2n-1}\rangle$ converges.

$\therefore$

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1. (c) Show that the $\sin x$ is uniformly continuous on $[0,\infty)$. (10)

let $f(x) = \sin x \quad \forall x \in [0,\infty)$

Now, for any $c \in [0,\infty)$

$$|f(x) - f(c)| = |\sin x - \sin c|$$

$$= \left|2 \cdot \sin\left(\frac{x-c}{2}\right) \cdot \cos\left(\frac{x+c}{2}\right)\right|$$

$$\leq \left|2 \sin\left(\frac{x-c}{2}\right)\right| \quad \because (|\cos\theta| \leq 1)$$

and we know that, $(\sin\theta \leq \theta)$

$$\therefore |f(x)-f(c)| \leq 2\left|\frac{x-c}{2}\right| = |x-c|$$

$$\Rightarrow |f(x)-f(c)| \leq |x-c| \;\&$$

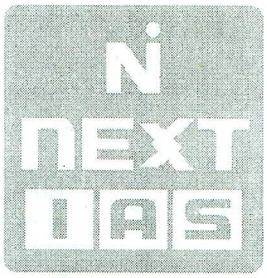
$$\Rightarrow |f(x)-f(c)| \leq \epsilon \text{ whenever, } |x-c| \leq \epsilon$$

$\therefore \forall \epsilon > 0, \exists \delta = \epsilon$ (dependent only on ϵ & not c)

s.t. $|f(x)-f(c)| < \epsilon$ whenever $|x-c| < \delta$

$\therefore f(x)$ is uniformly continuous on $[0,\infty)$

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1. (d) Let $f(z)$ be entire function satisfying $|f(z)| \leq k|z|^2$ for some positive constant k and all z. Show that $f(z) = az^2$ for some constant a. (10)

Let $f(z)$ be entire function

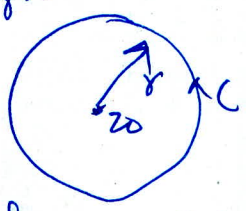
and $|f(z)| \leq k|z|^2$, $k>0$, $\forall z \in \mathbb{C}$

To show: $f(z) = az^2$, a is const.

We know that, By Cauchy's Inequality theorem

if $|f(z)| \leq M$ on C then $|f^{(n)}(z_0)| \leq \frac{M n!}{r^n}$

when f is analytic inside and on C



$\therefore$ choosing any C with radius r & ~~any $z_0 \in \mathbb{C}$~~ $z_0 = 0$

$$|f^{(n)}(z_0)| \leq \frac{k|z|^2 n!}{r^n}$$

$$= \frac{k|z|^2 n!}{r^n} \qquad |z| = r \text{ as } z_0 = 0$$

$$= \frac{k n!}{r^{n-2}}.$$

Putting $n = 3$

$$|f^{(3)}(0)| \leq \frac{kn!}{r}$$

Now, we can choose r as large as possible

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as f is entire function

$\therefore$ as $r \longrightarrow \infty$

$$|f^{(3)}(z_0)| \leq 0 \Rightarrow f^{(3)}(0) = 0$$

Ily, for any $z_0 \in \mathbb{C}$.

$$|f^{(3)}(z_0)| \leq \frac{k|z|^2 3!}{r^n}$$ and max^m value of $|z|$ can be at C i.e. $|z| = r$

$\therefore \forall z_0 \in C \quad |f^{(3)}(z_0)| \leq 0$

$\Rightarrow f^{(3)}(z_0) = 0 \quad \forall z_0 \in \mathbb{C}$

$\Rightarrow f(z_0)$ is polynomial of second degree.

i.e. $f(z) = az^2 + bz + c$

$\therefore |f(0)| \leq k|0|^2 = 0 \Rightarrow f(0) = 0$

$\Rightarrow \boxed{c = 0}$

$|f(z)| \leq k|z|^2$, Now by Cauchy's general Integral theorem

~~$\Rightarrow |z||az+b| \leq k|z|^2$~~

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

$$|f'(z_0)| = \left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz\right| \leq \frac{1}{2\pi}\oint_C \frac{|f(z)|}{|z-z_0|^2}dz.$$

$|f'(0)| \leq \frac{1}{2\pi} k\oint_C dz = 0$ ($\because C$ is closed).

$\Rightarrow f'(0) = 0.$

$\Rightarrow 2a(0) + b = 0 \Rightarrow \boxed{b = 0}$

$\therefore \boxed{f(z) = az^2}$

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1. (e) Show that if the optimal value of an objective function is attained at more than one extreme points then every convex combination of these extreme points also gives the optimal value of the objective function. (10)

Let w be objective function of a n-dimensional space within Convex boundary C

Let optimal value is attained at more than one point say X_1, X_2

where $X_1 = (x_1, x_2 \cdots x_n)$ & $X_2 = (y_1, y_2 \cdots y_n)$

To show: every Convex combination of X_1, X_2 gives optimal value

$\therefore$ w is optimal at X_1 & X_2

$\therefore$ $w(X_1) \geq w(X)$ $\forall$ $X \in$ Convex Set C

& $w(X_2) \geq w(X)$ $\forall X \in C$

Now any convex combination by $\frac{\lambda X_1 + X_2}{\lambda + 1}$

then $w\left(\frac{\lambda X_1 + X_2}{\lambda + 1}\right)$

$= w\left(\frac{\lambda x_1 + y_1}{\lambda + 1}, \frac{\lambda x_2 + y_2}{\lambda + 1}, \cdots \frac{\lambda x_n + y_n}{\lambda + 1}\right)$

$\geq \frac{\lambda w(X_1)}{\lambda + 1} + \frac{w(X_2)}{\lambda + 1}$

$\geq \frac{\lambda w(X)}{\lambda + 1} + \frac{w(X)}{\lambda + 1}$ $\forall X \in C$

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$\Rightarrow w\left(\frac{\lambda x_1 + x_2}{\lambda+1}\right) \geq \frac{\lambda+1}{\lambda+1} w(x) \quad \forall x \in C$

$\Rightarrow w\left(\frac{\lambda x_1 + x_2}{\lambda+1}\right) \geq w(x) \quad \forall x \in C$

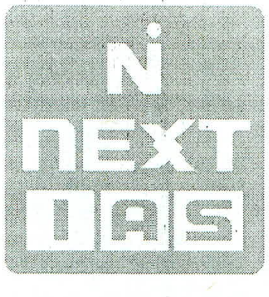
$\therefore \frac{\lambda x_1 + x_2}{\lambda+1}$ is also a optimal solution

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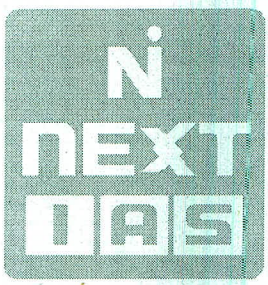
2. (a) Show that the polynomial $25x^4 + 9x^3 + 3x + 3$ is irreducible over the field of rational numbers. (15)

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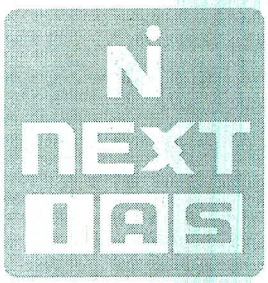


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2. (b) Prove that a bounded function f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there exist a partition P of $[a, b]$ such that $U(P,f) - L(P,f) < \epsilon$. (15)



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2. (c) Consider the following LPP,

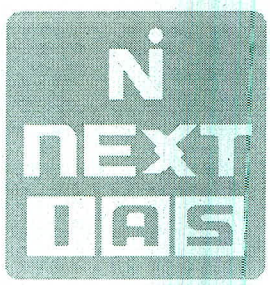
Maximise: $2x_1 + 4x_2 + 4x_3 - 3x_4$

Subject to $x_1 + x_2 + x_3 = 4$

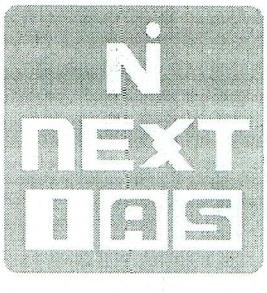
$x_1 + 4x_2 + x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

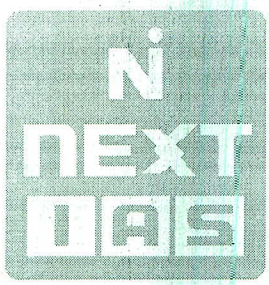
Use the dual problem to verify that the basic solution (x_1, x_2) is not optimal. (20)



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3. (a) Let N be a normal subgroup of a group G. Show that G/N is abelian if and only if for all $x, y \in G$, $xyx^{-1}y^{-1} \in N$. (20)

let N be Normal Subgroup of G

To prove G/N is abelian $\iff \forall x,y \in G \; xyx^{-1}y^{-1} \in N$

(I) let G/N is abelian

then $\forall Nx, Ny \in G/N \; , x, y \in N$

$(Nx)(Ny) = (Ny)(Nx)$

$\Rightarrow Nxy = Nyx$

$\Rightarrow Nxyx^{-1}y^{-1} = N$

$\Rightarrow \boxed{xyx^{-1}y^{-1} \in N} \quad \forall x, y \in G$

(II) let $\forall x, y \in G, \; xyx^{-1}y^{-1} \in N$

Now to prove G/N is abelian

let $Nx, Ny \in G/N \; \forall x, y \in G$

Now $\forall x, y \in G \quad xyx^{-1}y^{-1} \in N$

$\Rightarrow Nxyx^{-1}y^{-1} = N$

$\Rightarrow Nxy = Nyx$

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$\Rightarrow Nx Ny = Ny Nx$ ($\because$ N is Normal Subgroup) ($\because xN = Nx$)

$\Rightarrow (Nx)(Ny) = (Ny)(Nx)$

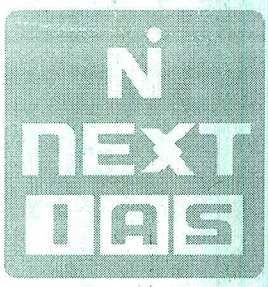
$\Rightarrow G/N$ is abelian

$\therefore G/N$ is abelian if & only if

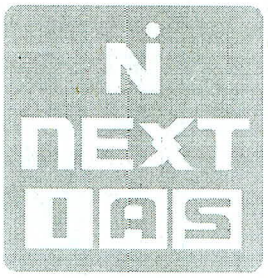
$\forall x, y \in G$, $xyx^{-1}y^{-1} \in N$

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3. (b) Let $\lim_{n\to\infty} a_n = l$, then prove that $\lim_{n\to\infty} \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n} = l$. (15)

Given that: $\lim_{n\to\infty} a_n = l$

Now, to prove: $\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots a_n}{n} = l$

$\because \lim_{n\to\infty} a_n = l$

$\therefore \forall \epsilon > 0$, $\exists\ m \in N$ s.t $\forall n \geq m$

$|a_n - l| < \epsilon$

Now, Consider

$$\left| \frac{a_1 + a_2 + \cdots a_n}{n} - l \right| = \left| \frac{(a_1 - l) + (a_2 - l) + \cdots (a_n - l)}{n} \right|$$

$$\leq \left| \frac{a_1 - l}{n} \right| + \left| \frac{a_2 - l}{n} \right| + \cdots \left| \frac{a_n - l}{n} \right|$$

Now, for any $\epsilon > 0$, we know $\exists m$ s.t

$|a_n - l| < \epsilon \quad \forall n \geq m$

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3. (c) State Cauchy's Residue Theorem. Using it to evaluate the integral

$$\int_C \frac{e^z+1}{z(z+1)(z-i)^2}dz, \quad C:|z|=2. \qquad (15)$$

Cauchy's Residue theorem:- If C be a closed curve and function $f(z)$ is analytic everywhere on C and inside C except at some point $z_1, z_2 \cdots z_n$

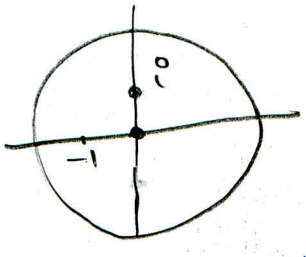
then $\int_C f(z)\,dz = 2\pi i\left(\text{residue}(z_1)+\text{residue}(z_2)\cdots \text{res}(z_n)\right)$

where residue (z_1) is coefficient of $\frac{1}{z-z_1}$ in Laurent's series expansion about z_1.

Let $I = \int_C \frac{e^z+1}{z(z+1)(z-i)^2}dz \quad C:|z|=2$

Singularity of $f(z) = \frac{e^z+1}{z(z+1)(z-i)^2}$.

$z=0, -1, i$



residue of $f(z)$ at $(z=0) = \lim_{z\to 0} z f(z)$ (∵ Simple pole)

$$= \lim_{z\to 0} \frac{e^z+1}{(z+1)(z-i)^2} = -2$$

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Residue of $f(z)$ at $(z=-1) = \lim_{z\to -1} (z+1) f(z)$ (Simple pole)

$$= \lim_{z\to -1} \frac{e^z+1}{z(z-i)^2}$$

$$= \frac{1+e^{-1}}{(-1)(-1-i)^2} = \frac{1+e^{-1}}{(-1)(1+(-1)+2i)}$$

$$= \frac{1+e^{-1}}{-2i} = i\left(\frac{1+e^{-1}}{2}\right)$$

Residue of $f(z)$ at $(z=i)$

$$= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 f(z)\right] \quad \text{(double pole)}$$

$$= \lim_{z\to i} \frac{d}{dz}\left[\frac{e^z+1}{z(z+1)}\right]$$

$$= \lim_{z\to i} \frac{e^z(z^2+z) - (e^z+1)(2z+1)}{z^2(z+1)}$$

$$= \lim_{z\to i} \frac{e^z(z^2+z-2z-1) - (2z+1)}{z^2(z+1)}$$

$$= e^i\left(\frac{-1+i-2i-1) - (2i+1)}{(-1)(1+i)}\right)$$

$$= e^i\left(\frac{-2-i) - (2i+1)}{-(1+i)}\right)$$

$$= -e^i\left(\frac{(2+i)(1-i) - (1+2i)(1-i)}{-(2)}\right)$$

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$$= e^{i}\left(3 - i\right) + \left(3 + i\right)$$

$$+2$$

$$= \frac{e^{i}(3-i) + (3+i)}{2}$$

Now By residue theorem,

$$\oint_C f(z)\,dz = 2\pi i\left(-2 + i\left(\frac{1+e^{-1}}{2}\right) + \frac{3+i}{2} + \frac{e^{i}(3-i)}{2}\right)$$

$$= 2\pi i\left(-\frac{1}{2} + \frac{i(2+e^{-1})}{2} + (\cos 1 + i\sin 1)\frac{(3-i)}{2}\right)$$

~~$= \pi i\left(-\frac{1}{2} + \frac{i(2+e^{-1})}{2}\right) + e$~~

$$\oint_C f(z)\,dz = \pi i\left(-1 + i(2+e^{-1}) + e^{i}\frac{(3-i)}{2}\right)$$

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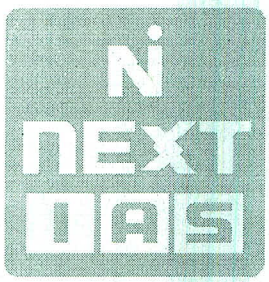
4. (a) Show that $Z\left[\sqrt{2}\right]$ is a Euclidean domain. (20)

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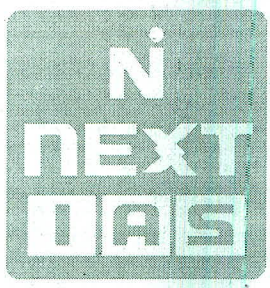
4. (b) Prove that if the convex set of feasible solutions of $AX = B, X \geq 0$ is convex bounded polyhedron then atleast one extreme point gives an optimal solution of the LPP. (15)

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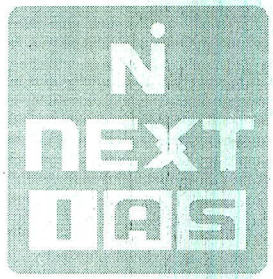


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4. (c) Classify the singular point $z = 0$ of the function $f(z) = \dfrac{e^z}{z - \sin z}$ and obtain the principal part of its Laurent Series expansion. (15)

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SECTION-B

5. (a) Solve: $\left(y^2+z^2-x^2\right)p-2xyq+2xz=0.$ (10)

Given P.d.e: $(y^2+z^2-x^2)p-2xyq=-2xz$

On comparing with $Pp+Qq=R$

$P=y^2+z^2-x^2 \quad Q=-2xy \quad R=-2xz$

Lagrange's Auxillary equation

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2xz}$$

Consider last 2 eq^n →

$$\frac{dy}{y}=\frac{dz}{z}$$

$$\Rightarrow \log y-\log z=\log q$$

$$\Rightarrow \boxed{y/z=q}$$

Consider multiplier x, y, z and $0, 1, 0$

$$\frac{x\,dx+y\,dy+z\,dz}{xy^2+xz^2-x^3-2xy^2-2xz^2}=\frac{dy}{-2xy}$$

$$\frac{x\,dx+y\,dy+z\,dz}{-x(x^2+y^2+z^2)}=\frac{dy}{-2xy}$$

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$$\frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2} = \frac{dy}{y}$$

$$\Rightarrow \log\left(\frac{x^2+y^2+z^2}{y}\right) = \log c_2$$

$$\Rightarrow \frac{x^2+y^2+z^2}{y} = c_2$$

∴ general solution of give p.d.e is

$$\boxed{\phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{y}\right) = 0}$$

where ϕ is an bitrary function

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5. (b) If $u = u(y-z,\ z-x,\ x-y)$, then find the value of $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$. (10)

given: $u = u(y-z,\ z-x,\ x-y)$

to find: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

let $a = y-z$, $b = z-x$ $c = x-y$

so that, $u = u(a, b, c)$

Now by chain rule of differentiation

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial x} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial x}$$

$$= \frac{\partial u}{\partial a}\cdot 0 + \frac{\partial u}{\partial b}(-1) + \frac{\partial u}{\partial c}(1) \quad -(I)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial y} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial y}$$

$$= \frac{\partial u}{\partial a} - \frac{\partial u}{\partial c} \quad -(II)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial z} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial z} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial z}$$

$$= -\frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} \quad -(III)$$

On adding eq (I), (II), (III)

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$$

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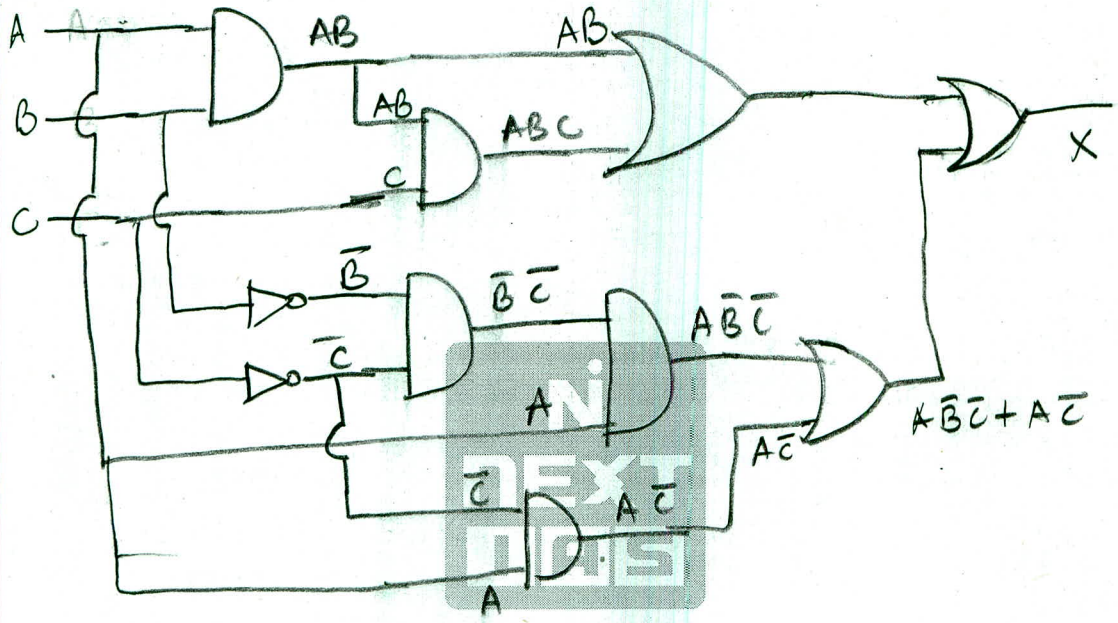
5. (c) Given the Boolean Expression $X = AB + ABC + A\bar{B}\bar{C} + A\bar{C}$ (10)

(i) Draw the logical diagram for the expression

(ii) Minimize the expression

(iii) Draw the logical diagram for the reduced expression

(i) $X = AB + ABC + A\bar{B}\bar{C} + A\bar{C}$



(ii) $X = AB + ABC + A\bar{B}\bar{C} + A\bar{C}$

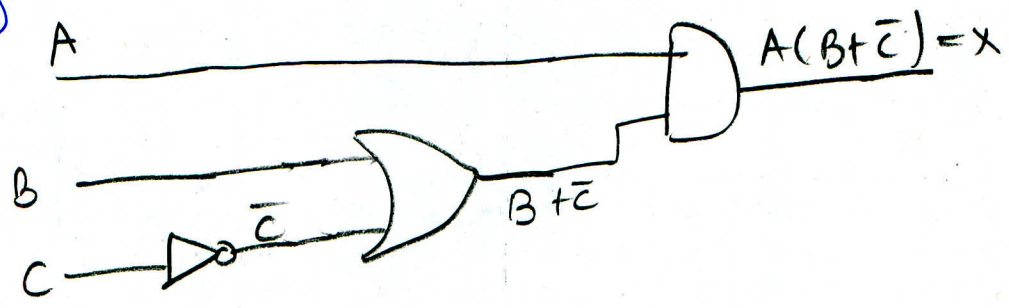
$= AB(1+C) + A\bar{C}(1+\bar{B})$ (distribution)

$= AB + A\bar{C}$ (dominance $1+X=1$)

$= A(B+\bar{C})$ distribution

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(iii)



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5. (d) Given the velocity potential $\phi=\frac{1}{2}\log\left(\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right)$, determine the streamlines. (10)

Given velocity potential $\phi=\frac{1}{2}\log\left(\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right)$

to find Streamlines ψ

we know that, stream line ψ & ϕ follow Cauchy reiman equation →

$\therefore$ ⊛ $\phi_x=\psi_y$ & $\phi_y=-\psi_x$

$$\psi_y=\phi_x=\frac{1}{2}\cdot\frac{(2(x+a))}{(x+a)^2+y^2}-\frac{1}{2}\cdot\frac{2(x-a)}{(x-a)^2+y^2}$$

$$=\frac{x+a}{(x+a)^2+y^2}-\frac{x-a}{(x-a)^2+y^2}$$

$$\& \ \psi_x=-\phi_y=-\frac{1}{2}\left[\frac{2y}{(x+a)^2+y^2}-\frac{2y}{(x-a)^2+y^2}\right]$$

$$=\frac{y}{(x-a)^2+y^2}-\frac{y}{(x+a)^2+y^2}$$

$\therefore$ ⊛ $\because$ ψ follow laplace eq^n

$\therefore$ $\psi_x\,dx+\psi_y\,dy$ is exact

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$$\therefore \psi = \int \psi_x \, dx + \int_{\text{only } y\text{-term}} \psi_y \, dy$$

$$= \int \frac{y}{(x-a)^2+y^2} \, dx - \frac{y}{(x+a)^2+y^2} \, dy$$

$$= \tan^{-1}\left(\frac{x-a}{y}\right) - \tan^{-1}\left(\frac{x+a}{y}\right) + C$$

$$= \tan^{-1}\left(\frac{-2a/y}{1+\frac{x^2-a^2}{y^2}}\right) + C$$

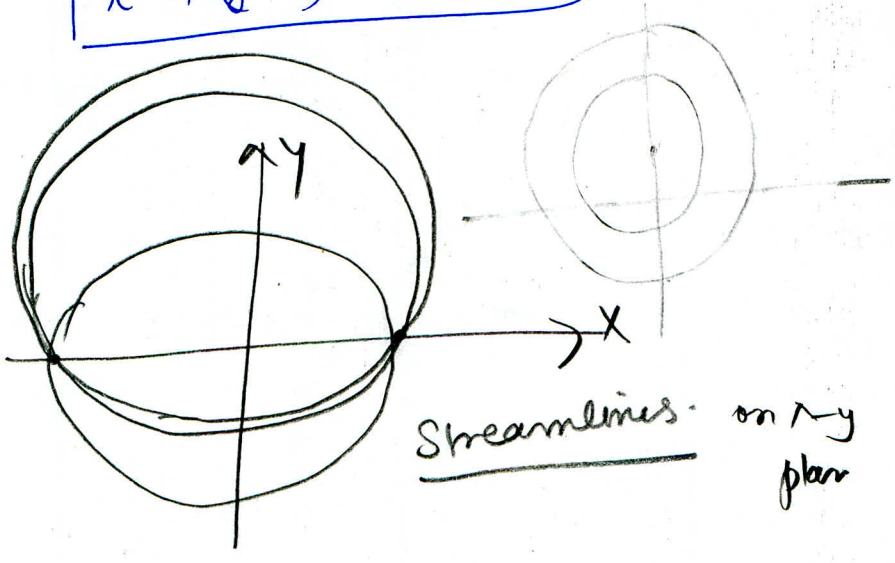
$$= \tan^{-1}\left(-\frac{2ay}{x^2+y^2-a^2}\right) + C$$

$\therefore$ Streamline are given by

$$\frac{y}{x^2+y^2-a^2} = \frac{1}{c}$$

or $x^2+y^2-ay = a^2$

or $\boxed{x^2+(y-c)^2 = a^2+c^2}$

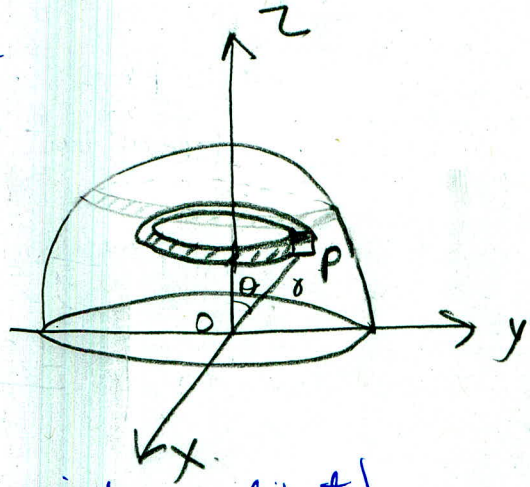


Streamlines on x-y plan

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5. (e) Calculate the moment of inertia of a solid uniform hemisphere $x^2+y^2+z^2=a^2, z\geq 0$ with mass m about the OZ-axis. (10)

Solid uniform hemisphere as shown with radius = a & mass m.



Consider an element ring formed by revolving element at P(r, θ) (Spherical Coordinate) & dimension δr & rδθ

Radius of element ring = $r\sin\theta$

Mass of element $dm = \frac{M}{\frac{2}{3}\pi a^3} \times (2\pi r\sin\theta) \times (\delta r \times r\delta\theta)$

$= \frac{3M}{2\pi a^3} 2\pi r^2 \sin\theta \, dr \, d\theta$

$= \frac{3M}{a^3} r^2 \sin\theta \, \delta r \, \delta\theta$

Now moment due to element ring

$dI = \int dm (r\sin\theta)^2$

$\therefore I = \int_{r=0}^{a}\int_{\theta=0}^{\pi/2} dm \, r^2 \sin^2\theta$

$= \int_{r=0}^{a}\int_{\theta=0}^{\pi/2} \frac{3M}{a^3} r^4 dr \cdot \sin^3\theta \, d\theta$

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$$= \frac{3M}{a^3} \int_{r=0}^{a} r^4 \, dr \times \int_{\theta=0}^{\pi/2} \sin^3\theta \, d\theta$$

$$= \frac{3M}{a^3} \left(\frac{a^5}{5}\right) \int_{\theta=0}^{\pi/2} B\left(\frac{3+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{3Ma^2}{5} \cdot \frac{\Gamma(2)\,\Gamma(1/2)}{\Gamma(2+1/2)}$$

$$= \frac{3Ma^2}{5} \cdot \frac{\Gamma(1/2)}{(1+1/2)(1/2)\Gamma(1/2)}$$

$$= \frac{3Ma^2}{5} \cdot \frac{4}{3} = \boxed{\frac{4Ma^2}{5}}$$

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6. (a) Solve by Charpit's method $pxy + pq + qy = yz$. (15)

let given diff. eq^n →

$$f(x,y,z,p,q) = pxy + pq + qy - yz = 0 \quad -(1)$$

Now, Charpit's Auxilary equation →

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\frac{dx}{-xy-q} = \frac{dy}{-p-y} = \frac{dz}{-pxy-pq} = \frac{dp}{py+p(-y)} = \frac{dq}{px+q} = \frac{dq}{-z-qy}$$

$$-pq-yq$$

Consider 4th eq $\frac{dp}{py-py} = \frac{dp}{0}$

$\Rightarrow dp = 0$

$\Rightarrow p = a$ (some arbitrary constant) $-(11)$

Substitute $p=a$ in eq (1)

$$axy + aq + qy - yz = 0$$

$$q(a+y) + axy - yz = 0$$

$$\Rightarrow q = \frac{yz - axy}{a+y} = \frac{y(z-ax)}{a+y} \quad -(111)$$

Now putting (e1) (11) & (111) in →

$$dz = p\,dx + q\,dy$$

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$$dz = a\,dx + \frac{y(z-ax)}{a+y}\,dy$$

$$dz - a\,dx = \frac{y}{a+y}(z-ax)\,dy$$

$$\Rightarrow \frac{d(z-ax)}{z-ax} = \frac{y\,dy}{a+y}$$

On Integrating both side →

$$\log(z-ax) = \int 1 - \frac{a}{a+y}\,dy$$

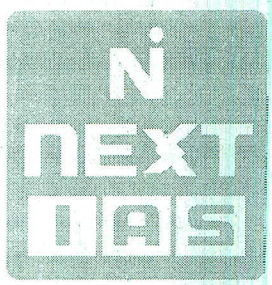
$$= y - a\log(a+y) + \log b$$

$$\log(z-ax) = \log e^{y} - \log(a+y)^{a} + \log b$$

$$\Rightarrow \boxed{z - ax = \frac{be^{y}}{(y+a)^{a}}}$$

Complete solution of eq (1).

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6. (b) The equation $x^2 + ax + b = 0$ has two real roots α and β. Show that the iteration method

(i) $x_{k+1} = -\dfrac{ax_k + b}{x_k}$ is convergent near $x = \alpha$ if $|\alpha| > |\beta|$.

(ii) $x_{k+1} = -\dfrac{b}{x_k + a}$ is convergent near $x = \alpha$ if $|\alpha| < |\beta|$.

(iii) $x_{k+1} = -\dfrac{x_k^2 + b}{a}$ is convergent near $x = \alpha$ if $2|\alpha| < |\alpha + \beta|$. (15)

Given that $x^2+ax+b=0$ has two real root α & β.

We know that, if iteration formula is $x_{k+1} = \phi(x_k)$, then it is Convergent if $|\phi'(x_k)| < 1$

(i) $x_{k+1} = -\dfrac{ax_k+b}{x_k}$

$\phi(x_k) = -\left(\dfrac{ax_k+b}{x_k}\right) = -a - \dfrac{b}{x_k}$

$|\phi'(x_k)| = \left|\dfrac{b}{x_k^2}\right| < 1$

$\Rightarrow |x_k^2| > |b|$

if it converge about $x_k = \alpha$ & b = product of root

$\Rightarrow b = \alpha\beta$.

$\therefore |\alpha|^2 > |\alpha||\beta|$

$\Rightarrow \boxed{|\alpha| > |\beta|}$

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(ii) $x_{k+1} = \dfrac{-b}{x_k + a}$.

$\phi(x_k) = \dfrac{-b}{x_k + a}$

$\phi'(x_k) = \dfrac{b}{(x_k+a)^2}$

$|\phi'(x_k)| < = \left|\dfrac{b}{(x_k+a)^2}\right| < 1$ (for convergence)

$\Rightarrow (x_k + a)^2 > b$

we know sum of roots $= \alpha + \beta = (-a)$

$\therefore$ if x_k converge around α

$|\alpha - \alpha - \beta|^2 > |\alpha||\beta|$

$\Rightarrow$ $|\beta| > |\alpha|$

(iii) $x_{k+1} = -\dfrac{x_k^2 + b}{a}$

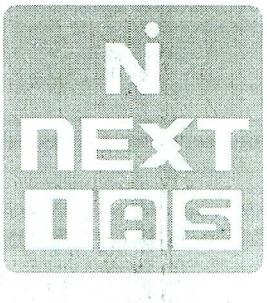
$\phi(x_k) = -\dfrac{x_k^2 + b}{a}$

$\phi'(x_k) = -\dfrac{2x_k}{a}$

$|\phi'(x_k)| = \left|\dfrac{2x_k}{a}\right| < 1$ (for convergence)

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$\Rightarrow \boxed{|2\alpha| < |\alpha+\beta|}$



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6. (c) A steady inviscid incompressible fluid flow has a velocity field $u = fx, v = -fy, w = 0$, where f is a constant. Derive an expression for the pressure field $p(x,y,z)$ if the pressure $p(0,0,0) = p_0$ and $\boldsymbol{F} = -gz\hat{k}$. (20)

Given that: fluid is Steady, inviscid, incompressible

and velocity $u = fx \quad v = -fy \quad w = 0$

$q = u\hat{i} + v\hat{j} + w\hat{k}$ / f - constant

$p(0,0,0) = p_0$ & $\bar{F} = -gz\hat{k}$.

Motion clearly follows equation of continuity

and equation of Motion (Euler's) is given by $\rightarrow$

ρ is density.

$$\frac{Dq}{Dt} = F - \frac{\nabla p}{\rho}$$

$$\frac{\partial q}{\partial t} + (q \cdot \nabla) q = F - \frac{\nabla p}{\rho}$$

$$u\frac{\partial u}{\partial x}\hat{i} + v\frac{\partial v}{\partial y}\hat{j} + w\frac{\partial w}{\partial z}\hat{k} = F - \frac{\nabla p}{\rho}$$

as $\left(\frac{\partial q}{\partial t} = 0 \text{ in}\right)$ (Steady flow)

$$fx(f)\hat{i} + (-fy)(-f)\hat{j} + 0 = -gz\hat{k} - \frac{\nabla p}{\rho}$$

$$\Rightarrow -\frac{\nabla p}{\rho} = f^2 x\hat{i} + f^2 y\hat{j} + gz\hat{k}$$

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$$-\nabla p = \rho f^2 x\hat{i} + \rho f^2 y\hat{j} + \rho g z\hat{k}$$

$$-\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} - \frac{\partial p}{\partial z}\hat{k} = \rho f^2 x\hat{i} + \rho f^2 y\hat{j} + \rho g z\hat{k}$$

On comparing component of $\hat{i}$, $\hat{j}$ & $\hat{k}$.

$$-\frac{\partial p}{\partial x} = \rho f^2 x \qquad -\frac{\partial p}{\partial y} = \rho f^2 y$$

and $$-\frac{\partial p}{\partial z} = \rho g z$$

Consider $$\frac{\partial p}{\partial x} = -\rho f^2 x$$

$$\Rightarrow p = -\frac{\rho f^2}{2} x^2 + f_1(y,z) \qquad \text{—(I)}$$

Consider, $$\frac{\partial p}{\partial y} = -\rho f^2 y$$

$$p = -\frac{\rho f^2}{2} y^2 + f_2(x,z) \qquad \text{—(II)}$$

Consider, $$\frac{\partial p}{\partial z} = -\rho g z$$

$$\Rightarrow p = -\frac{\rho g}{2} z^2 + f_3(x,z) \qquad \text{—(III)}$$

On Combining eq (I), (II) & (III)

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$$p = -\frac{\rho f^2 x^2}{2} - \frac{\rho f^2 y^2}{2} - \frac{\rho g}{2} z^2 + C$$

$\because$ $p(0,0,0) = p_0$

$\therefore$ $C = p_0$

$$\Rightarrow \boxed{p = -\frac{\rho f^2 x^2}{2} - \frac{\rho f^2 y^2}{2} - \frac{\rho g}{2} z^2 + p_0}$$

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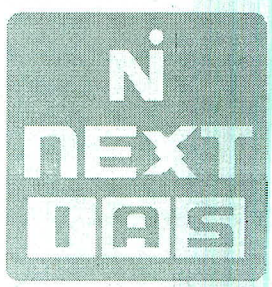
7. (a) Find the equations of the system of curves on the cylinder $2y = x^2$ orthogonal to its intersections with the hyperboloids of one-parameter system $xy = z + c$. (15)

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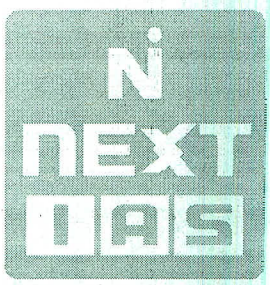
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7. (b) For an integral $\int_{-1}^{1} f(x)dx$, show that two-point Gauss Quadrature Rule is given by $\int_{-1}^{1} f(x)dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$. Using this rule, estimate $\int_{2}^{4} 2xe^{x}\,dx$.

(20)



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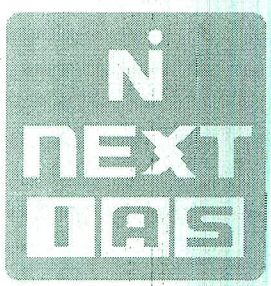


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7. (c) Consider a system with two degrees of freedom for which $V = q_1^2 + 3q_1q_2 + 4q_2^2$. Find the equilibrium position and determine if the equilibrium position is stable. (15)



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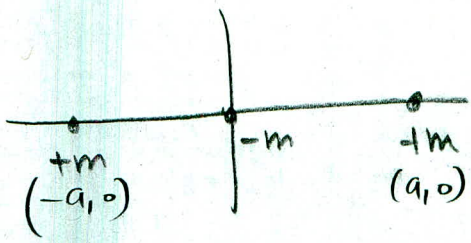


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8. (a) Find the stream function of the two-dimensional motion due to two equal sources and an equal sink situated midway between them. (15)

let two equal source of strength 'm' be situated at $(a,0)$ & $(-a,0)$ and an equal sink of strength '$-m$' at $(0,0)$



then, Complex potential due to the system

$$\omega = -m\log(z-a) - m\log(z+a) + m\log z$$

$$\omega = m\log\frac{z}{z^2-a^2}$$

Now, Stream function is given by imaginary component of ω.

$$\omega = m\log\frac{x+iy}{} - m\log(x-a+iy) - m\log(x+a+iy$$

then $\psi = m\tan^{-1}\left(\frac{y}{x}\right) - m\tan^{-1}\frac{y}{x-a} - m\tan^{-1}\frac{y}{x+a}$

$$= m\tan^{-1}\left(\frac{y}{x}\right) - m\left(\tan^{-1}\left(\frac{\frac{y}{x-a}+\frac{y}{x+a}}{1-\frac{y^2}{x^2-a^2}}\right)\right)$$

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$$= m\tan^{-1}\left(\frac{y}{x}\right) - m\tan^{-1}\left(\frac{yx + ya + yx - ay}{x^2 - a^2 - y^2}\right)$$

$$= m\tan^{-1}\left(\frac{y}{x}\right) - m\tan^{-1}\left(\frac{2xy}{x^2 - y^2 - a^2}\right)$$

$$\psi = m\tan^{-1}\left(\frac{\frac{y}{x} - \frac{2xy}{x^2 - y^2 - a^2}}{1 + \frac{2xy^2}{x\cdot(x^2 - y^2 - a^2)}}\right)$$

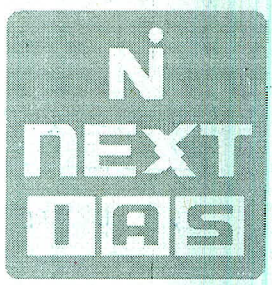
$$= m\tan^{-1}\left(\frac{yx^2 - y^3 - ya^2 - 2x^2y}{x^3 - xy^2 - xa^2 + 2xy^2}\right)$$

$$= m\tan^{-1}\left(\frac{-y^3 - x^2y - ya^2}{x^3 + xy^2 - xa^2}\right)$$

$$\boxed{\psi = m\tan^{-1}\left(-\frac{y}{x}\left(\frac{y^2 + x^2 + a^2}{x^2 + y^2 - a^2}\right)\right)}$$

Stream function ψ.

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8. (b) Write down the algorithm for solving the equation: $xe^x - 1 = 0$ by bisection method, correct to 4 decimal places. (15)

1. Define $f(x) = xe^x - 1$
2. Read a, b
3. while (f(a) * f(b) > 0)
4. Print " Entered value cannot Calculate root "
5. Read a, b
6. End while
7. mid ← $\frac{a+b}{2}$
8. while (f(mid) ≠ 0 and |a − b| > 10^{-5})
9. mid = $\frac{a+b}{2}$
10. if (~~f(a)=0~~ f(mid) = 0) then
11. break
12. else if (f(a) * f(mid) < 0) then
13. b ← mid
14. else
15. a ← mid
16. end if
17. end while.
18. ~~Print~~ if (f(mid) = 0) then
19. Print "found exact root = ", mid

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20. else
21. Print " root Correct upto 4 decimal plac ", mid
22. end if
23. Stop.

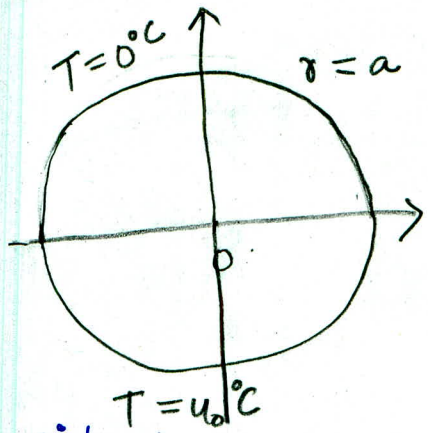
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8. (c) Find the steady state temperature in a circular plate of radius a which has one half of its circumference at $0°C$ and the other half at constant temperature $u_0 C$. (20)



let Circular plate of radius a be placed on Polar plane with Centre at O.

let Circumference above initial line is at $0°C$ ($\theta = 0$ to π)

and Circumfrence below initial line is $u_0 C$ i.e ($\theta = \pi$ to 2π)

let at steady state, temperature of any point (r, θ) of disc is given by $u(r, \theta)$

Now, we know on Steady State u will follow laplace equation. ie.

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{r \partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

in polar form

let u follow Separation of Variable method

$$u(r, \theta) = R(r)\,\Theta(\theta)$$

then $$R''\Theta + \frac{R'\Theta}{r} + \frac{R\Theta''}{r^2} = 0$$

$$\Rightarrow \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta}$$

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$$\frac{r^2R'' + rR'}{R} = -\frac{\theta''}{\theta} = a \text{ (const.)}$$

(Case I) if $a = 0$ then, $\theta'' = 0$

$\Rightarrow \theta = a\theta + b$ but $\theta(2\pi+\alpha) = \theta(\alpha)$

which does not satisfy

$\therefore$ Not a feasible solution

(Case II) if $a < 0$ then let $a = -k^2$.

$\theta'' - k^2\theta = 0$

$\Rightarrow \theta = a e^{k\theta} + b e^{-k\theta}$

which does not satisfy $\theta(2\pi+\alpha) = \theta(\alpha)$

$\therefore$ Not feasible solution

(Case III:) if $a < 0$ let $a = +k^2$

then $\theta'' + k^2\theta = 0$

$\Rightarrow \theta = a' \cos(k\theta) + b' \sin(k\theta)$

Now, $r^2R'' + rR' - k^2R = 0$

It is of homogenous form of euler.

given by $(r^2D^2 + rD - k^2)R = 0$

let $r = e^z$ i.e $z = \log r$

then,

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$r^2 D^2 = D_1(D_1 - 1)$

$rD = D_1$

$\therefore (D_1^2 - D_1 + D_1 - k^2) R = 0$

$(D_1^2 - k^2) R = 0$

$\Rightarrow$ Auxillary equation is ~~$m^2 + k^2 = 0$~~

~~$m = \pm ki$~~ $m^2 - k^2 = 0$

$m = \pm k$

$\therefore R = c_1 e^{kz} + c_2 e^{-kz}$

$R = c_1 r^k + c_2 r^{-k}$

$\because R(r=0)$ is well defined $\therefore c_2 = 0$

$\Rightarrow R = c_1 r^k$

$\therefore u(r,\theta) = R(r) \cdot \Theta(\theta)$

$= c_1 r^k (a' \cos(k\theta) + b' \sin(k\theta))$

$$u(r,\theta) = r^k (a' \cos k\theta + b' \sin k\theta)$$

Now $u(a,\theta) = \begin{cases} 0 & 0 < \theta < \pi \\ u_0 & \pi < \theta < 2\pi \end{cases}$

$\Rightarrow$ a Now, there can be many value of k.

$$u(r,\theta) = \sum_{n=1}^{\infty} r^n (E_n \cos n\theta + F_n \sin n\theta)$$

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at $u(a,\theta) = \sum a^n (E_n \cos n\theta + F_n \sin n\theta)$

$= \sum a^{2m+1} (E_n \cos(2m+1)\theta + F_n \sin(2m+1)\theta)$

$+ \sum a^{2m} (E_n \cos 2m$

Without loss of generality, we can assume Circonfre

follow, $u(a,\theta) = \begin{cases} 0 & -\pi/2 < \theta < \pi/2 \\ u_0 & \pi/2 < \theta < 3\pi/2 \end{cases}$

then $u(a,0) = \sum a^n (E_n) = 0$

$\Rightarrow E_n = 0$

$\therefore u(r,\theta) = \sum F_n r^n \sin n\theta$

$u(a,\theta) = \sum F_n a^n \sin n\theta = \begin{cases} 0 & -\pi/2 < \theta < \pi/2 \\ u_0 & \pi/2 < \theta < 3\pi/2 \end{cases}$

which is fourier Sine Series.

$F_n a^n = \frac{2}{\pi} \int_0^{\pi} u_0 \sin n\theta \, d\theta$

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SPACE FOR ROUGH WORK

$\forall \epsilon > 0, \exists m$

$|a_{2n-1} - a_{2m-1}| < \epsilon \quad \forall n \geq m$

$\forall \epsilon > 0 \exists p$

$|a_{2n} - a_{2p}| < \epsilon \quad \forall n \geq p.$

take $\max(m, p)$.

NEXT IAS