NEXT IAS

(To be filled by candidate)

Name : MADHUKANT T-I|2

Rol No. : MT24MATRA007 Date of Exam.: 26/Aug/2024

Mobile No. : ..... Email: .....

Exam Centre : Old Rajinder Nagar ☑ Bhopal ☐ Jaipur ☐
Mukherjee Nagar ☐ Prayagraj ☐ Online ☐

anubhav-2024

Paper-I	MATHEMATICS OPTIONAL
Dated : 26-08-2024	

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

*There are **EIGHT** questions divided in **Two sections** and printed in ENGLISH.*

*Candidate has to attempt **FIVE** questions in all.*

***Question No. 1 and 5** are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.*

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA) booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order, unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA) booklet must be clearly struck off.

(For filling by Examiners only)

Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	15			
2. (c)	20	20			
3. (a)	24	20			
3. (b)	28	15			
3. (c)	31	15			
4. (a)	34	15			
4. (b)	37	15			
4. (c)	40	20			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	15			
7. (b)	67	20			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	80	20			
Grand Total					

Remarks

Observations:

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SECTION-A

1. (a) Show that any two eigen vectors corresponding to the distinct eigenvalues of a unitary matrix are orthogonal. (10)

Let λ_1, λ_2 are two eigenvalues of matrix A and there corresponding eigen vector be X_1, X_2.

Also, $\lambda_1 \neq \lambda_2$ i.e. eigen values are distinct and A is unitary matrix i.e. $A^\theta A = I$.

$\therefore \quad AX_1 = \lambda_1 X_1 \quad \& \quad AX_2 = \lambda_2 X_2$

$(AX_1)^\theta = (\lambda_1 X_1)^\theta$

$\Rightarrow X_1^\theta A^\theta = \bar{\lambda}_1 X_1^\theta$

$\Rightarrow \boxed{X_1^\theta A^\theta X_2 = \bar{\lambda}_1 X_1^\theta X_2} \quad \text{— (1)}$

Consider $AX_2 = \lambda_2 X_2$

$A^\theta A X_2 = \lambda_2 A^\theta X_2$ (multiplying A^θ)

$X_2 = \lambda_2 A^\theta X_2$

$\therefore$ Putting in (1)

$X_1^\theta X_2 = \bar{\lambda}_1 \lambda_2 X_1^\theta X_2$

$\Rightarrow X_1^\theta X_2 (\bar{\lambda}_1 \lambda_2 - 1) = 0$

$\because \lambda_1 \neq \lambda_2 \Rightarrow \boxed{X_1^\theta X_2 = 0}$

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$\therefore$ X_1, X_2 are orthogonal.

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1. (b) Find the sum and product of the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{bmatrix}$. (10)

We know that sum of eigen value is given by trace (of matrix)

In this $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{bmatrix}$

trace $A = 1+1+3+5 = 10$

$\therefore$ **Sum of eigen value = 10**

product of eigen value $= (tr(A^2))^2 - tr(A^4)^2$

$$trace(A^2) = tr\left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 42 & & & \\ & 43 & & \\ & & 50 & \\ & & & 59 \end{bmatrix}$$

$$= 194$$

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$\therefore$ product of eigen value $= tr(A^2) - (tr(A))^2$

$= 194 - 100$

$= \boxed{94}$

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1. (c) If $u = z\sin\left(\frac{y}{x}\right)$; where $x = 3r^2 + 3s$, $y = 4r - 2s^3$, $z = 2r^2 - 3s^2$, then find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$. (10)

given: $u = z\sin\left(\frac{y}{x}\right)$, $x = 3r^2 + 3s$

$y = 4r - 2s^3$

$z = 2r^2 - 3s^2$

differentiating using chain rule →

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= \left(-\frac{yz}{x^2}\right)\cos\left(\frac{y}{x}\right)\times 6r + \frac{z}{x}\cos\left(\frac{y}{x}\right)\times 4 + \sin\left(\frac{y}{x}\right)\times 4r$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{z}{x}\cos\left(\frac{y}{x}\right)\left[4 - \frac{6yr}{x}\right] + 4r\sin\left(\frac{y}{x}\right)}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= \left(-\frac{yz}{x^2}\right)\cos\left(\frac{y}{x}\right)\times 3 + \frac{z}{x}\cos\left(\frac{y}{x}\right)\times(-6s^2) + \sin\left(\frac{y}{x}\right)\times(-6s)$$

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$$\frac{\partial u}{\partial s} = \frac{z}{x}\cos\left(\frac{y}{x}\right)\left[-6s^2 - 3\frac{y}{x}\right] - 6s\sin\left(\frac{y}{x}\right)$$

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1. (d) Evaluate: $\lim_{x\to\infty}\left\{\frac{\log x}{x}\right\}^{\frac{1}{x}}$. (10)

Let $y = \lim_{x\to\infty}\left(\frac{\log x}{x}\right)^{1/x}$

taking log on both side

$$\log y = \lim_{x\to\infty} \frac{1}{x}\left[\log(\log x) - \log x\right]$$

$$= \lim_{x\to\infty} \frac{\log(\log x)}{x} - \lim_{x\to\infty} \frac{\log x}{x}$$

Both are $\frac{\infty}{\infty}$ form

∴ we will use L'Hospital

$$\log y = \lim_{x\to\infty} \frac{\frac{1}{\log x}\times\frac{1}{x}}{1} - \lim_{x\to\infty} \frac{1/x}{1}$$

$$= 0$$

$$\Rightarrow y = e^0$$

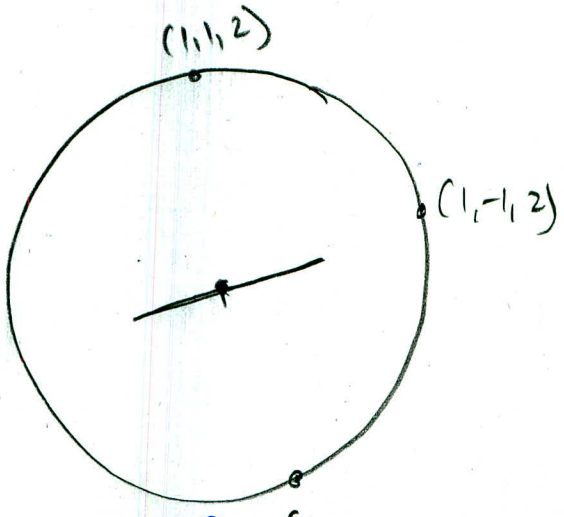
$$\Rightarrow \boxed{y = \lim_{x\to\infty}\left(\frac{\log x}{x}\right)^{1/x} = 1}$$

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1. (e) Find the equation of the sphere passing through the points (1,1,2), (1,−1,2) and having centre on the line $x+y-z-1=0=2x+y-z-2$. (10)



let $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ be equation of sphere

As it passes through $(1,1,2)$ and $(1,-1,2)$

$\therefore 6+2u+2v+4w+d=0$ — (I)

$6+2u-2v+4w+d=0$ — (II)

and centre passes through the line:

$x+y-z-1=0=2x+y-z-2$

$\therefore$ Centre is $(-u,-v,-w)$

$\therefore -u-v+w-1=0$ — (III)

$-2u-v+w-2=0$ — (IV)

eliminate d from (I) & (II)

$4v=0 \Rightarrow v=0$

Now $-u+w-1=0$

$-2u+w-2=0$

$u+1=0 \Rightarrow u=-1$

& $w=0$

$\therefore$ from eq (I) d will be

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$6 - 2 + d = 0$

$d = -4.$

$\therefore$ equation of sphere will be

$$x^2 + y^2 + z^2 - 2x - 4 = 0$$

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2. (a) Show that the linear transformation from IR^3 to IR^4 which is represented by the matrix $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$ is one to one. Find the basis for its image. (15)

let T be linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$

given a $[T] = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

let e_1, e_2, e_3 are standard basis of $\mathbb{R}^3$

then Now, let for any (x, y, z) $(\alpha, \beta, \gamma) \in \mathbb{R}^3$

$$T(x, y, z) = T(\alpha, \beta, \gamma)$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x+3y \\ y-2z \\ 2x+y+z \\ -x+y+2z \end{bmatrix} = \begin{bmatrix} \alpha+3\beta \\ \beta-2\gamma \\ 2\alpha+\beta+\gamma \\ -\alpha+\beta+2\gamma \end{bmatrix}$$

let $(x, y, z) \in \mathbb{R}^3$ s.t $T(x, y, z) = 0$

then $\begin{pmatrix} x+3y \\ y-2z \\ 2x+y+z \\ -x+y+2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ —①

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comparing 1st three eqⁿ →

$x + 3y = 0$
$y - 2z = 0$
$2x + y + z = 0$

} should satisfy $-x + y + 2z = 0$

above 3 eqⁿ can be written as $AX = 0$

where $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \end{pmatrix}$ & $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\therefore \det(A) = 1(1+2) - 3(4) = -9 \neq 0$

$\therefore X = (x \ y \ z)^T = (0 \ 0 \ 0)^T$ is only Solution and it also satisfy $-x + y + 2z = 0$

$\therefore \boxed{T(x, y, z) = 0 \Rightarrow (x, y, z) = 0}$

Now, let any $T(x, y, z) = T(\alpha, \beta, \gamma)$

$T((x, y, z) - (\alpha, \beta, \gamma)) = 0$

($\because$ T is linear transformation)

$\Rightarrow \boxed{(x, y, z) = (\alpha, \beta, \gamma)}$

$\therefore$ T is <u>one-one</u>

Image of T is given by linear span of

$T(\alpha_1), T(\alpha_2), T(\alpha_3)$

i.e. $\begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 2 \end{pmatrix}$

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Now, to check if above 3 are L. Independt

$$x\begin{pmatrix}1\\0\\2\\-1\end{pmatrix} + y\begin{pmatrix}3\\1\\1\\1\end{pmatrix} + z\begin{pmatrix}0\\-2\\1\\2\end{pmatrix} = 0$$

$$\begin{pmatrix}x+3y\\y-2z\\2x+y+z\\-x+y+2z\end{pmatrix} = 0$$

thus eqⁿ is same as recieved in eq (1)

∴ only possible solution is $(0,0,0)$

∴ given vector a Linearly Independt

∴ Basis for image of T is

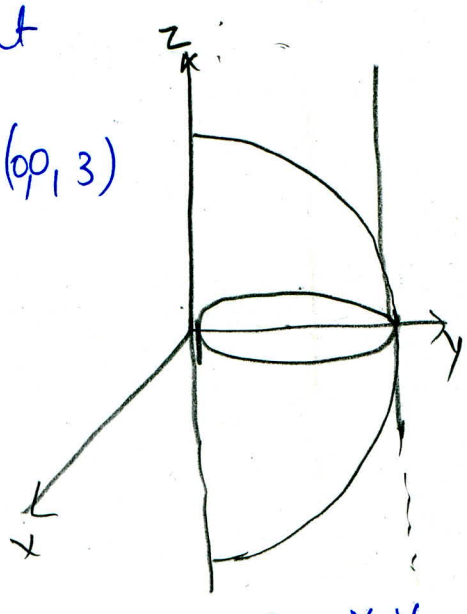
$$B = \{(1,0,2,-1)^T, (3,1,1,1)^T, (0,-2,1,2)^T\}$$

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2. (b) Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$. (15)

let O be Origin and point of Common surface be A i.e (0, 0, 3) as both passes through it.

Normal at any point of sphere will be $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$



& let R be region of cylinder projected on X-Y plan

then area of portion in cylinder

$$A = \iint_R dS = \iint_R \frac{dx\,dy}{1/\sqrt{3}} \qquad \left(\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{3}}\right)$$

$$= \sqrt{3} \iint_R dx\,dy$$

$$= \sqrt{3} \int_{y=0}^{3} \int_{x=-\sqrt{3y-y^2}}^{+\sqrt{3y-y^2}} dx\,dy$$

$$= \sqrt{3} \int_{y=0}^{3} 2\sqrt{3y-y^2}\,dy$$

$$= 2\sqrt{3} \int_{y=0}^{3} \sqrt{\frac{9}{4} - \left(y-\frac{3}{2}\right)^2}\,dy$$

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$$= 2\sqrt{3} \int_{y=0}^{3} \sqrt{y}\,\sqrt{3-y}\,dy$$

let $\frac{y}{3} = t$ $\quad dy = 3\,dt$

y	0	3
t	0	1

$$\Rightarrow A = 2\sqrt{3} \int_{0}^{1} \sqrt{3t}\,\sqrt{3-3t}\cdot 3\,dt$$

$$= 2\sqrt{3}\cdot\sqrt{3}\cdot\sqrt{3}\cdot 3 \int_{0}^{1} \sqrt{t}\,\sqrt{1-t}\,dt$$

$$= 18\sqrt{3}\; B(3/2, 3/2)$$

$$= 18\sqrt{3}\,\frac{\Gamma(3/2)\,\Gamma(3/2)}{\Gamma(3)}$$

$$= \frac{18\sqrt{3}}{2}\cdot\left(\tfrac{1}{2}\Gamma(1/2)\right)^2$$

$$= \frac{18\sqrt{3}}{8}\,\pi = \boxed{\frac{9\sqrt{3}}{4}\,\pi}$$

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2. (c) Prove that the three planes $2x + y + z = 3$, $x - y + 2z = 4$ and $x + z = 2$ form a triangular prism and find the area of the normal section of the prism. (20)

Given planes are $P_1: 2x+y+z=3$

$P_2: x-y+2z=4$

$P_3: x+z=2$

To prove: (i) P_1, P_2, P_3 form triangular prism

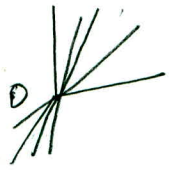
(ii) and find area of normal section of prism

On projecting planes to origin

$2x+y+z=0$

$x-y+2z=0$

$x+z=0$



Consider $\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 2(-1)-1(-1)+1(1)$

$= 0$

$\therefore$ there will be common line or a triangular prism as it projection on origin as ∞

Solution

Now, Common line given by P_1 and $P_2 \rightarrow$

$2x+y+z-3 = 0 = x-y+2z-4$

let d.r. will be (l, m, n) then

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$2l+m+n=0$

$l-m+2n=0$

$\frac{l}{3} = \frac{m}{-3} = \frac{n}{-3} \Rightarrow \frac{l}{1} = \frac{m}{-1} = \frac{n}{-1}$.

Now, clearly l, m, n are $\perp$ to P_3 as

$(l, m, n)\cdot(1, 0, 1) = 0$.

$\therefore$ line from $P_1 P_2$ is $\perp$ to P_3.

Now any point on common line $P_1 P_2$ is $\rightarrow$

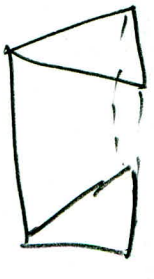
$2x+y+0-3=0$ (on taking $z=0$)

$x-y-4=0$

$x = 7/3 \quad y = -5/3$

Now, we will check if $(7/3, -5/3, 0)$ ~~satisfies~~ on P_3.

$\frac{7}{3} + 0 \neq 2 \Rightarrow$ common line $P_1 P_2$ does not lie on P_3 & thus, $P_1 P_2 P_3$ form a Prism.



Normal direction of normal section of prism will be $\parallel$ to d.r. of line b/w P_1, P_2. i.e $(1, -1, -1)$

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Now, intersection of Common line with $x-y$ plane i.e. $z=0$ is

for P_1-P_2 : $\left(\frac{7}{3}, -\frac{5}{3}, 0\right)$ (already found)

for P_2-P_3: $x-y=4$ $\Rightarrow y=-2$

$x=2$

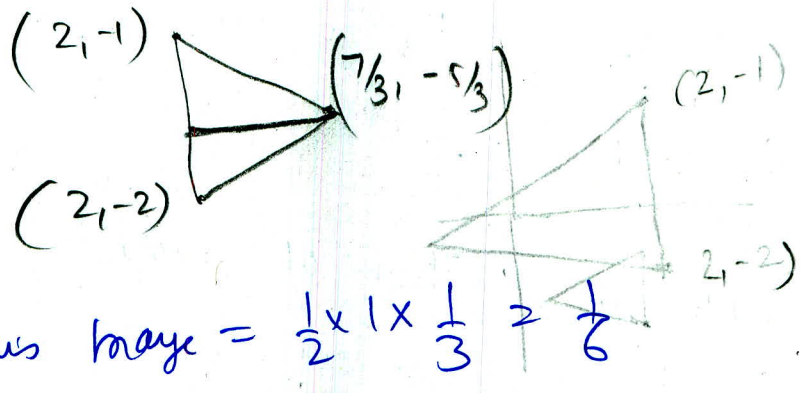
$\therefore (2,-2,0)$

for P_1-P_3: $2x+y=3$ $\Rightarrow y=-1$

$x=2$

$\therefore (2,-1,0)$

Now, area of triangle on $x-y$ plane formed by coordinate $\left(\frac{7}{3}, -\frac{5}{3}\right), (2,-2)$ & $(2,-1)$ is



Area of this traye $= \frac{1}{2} \times 1 \times \frac{1}{3} = \frac{1}{6}$

Now projection of arc on d.r. $(1,-1,-1)$

will be $\frac{1}{6} \times (\text{d.c. of } x-y \text{ plane}) \cdot \frac{(1,-1,-1)}{\sqrt{3}}$

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d.c of x-yplan = 0, 0, 1)

∴ projection of triangle on 1, 1, 1

will = $\left| \frac{1}{6} \times \frac{1}{\sqrt{3}} \right|$

⇒ Area of Normal section = $\boxed{\frac{1}{6\sqrt{3}}}$



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3. (a) Diagonalise the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ through an orthogonal transformation.

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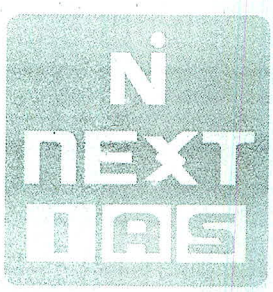
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3. (b) Given $xy = c$, where c is constant. Show that the minimum value of $(x + y)$ occurs, when $x = y$. Deduce that the minimum value of $a\cot\theta + b\tan\theta$ is $2\sqrt{ab}$, where a and b are constants. (15)

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3. (c) Prove that the centres of the spheres which touch the lines $y = mx,\ z = c;$ $y = -mx,\ z = -c$ lie upon the conicoid $mxy + cz(1 + m^2) = 0.$ (15)

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4. (a) State and prove Cayley Hamilton Theorem. (15)

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Cayley Hamilton theorem:

If A be a square matrix then it satisfy its characteristics eqn

i.e $\det(A - \lambda I) = 0$

let $f(\lambda) = 0$ represent characteristic eqn of A.

and $\lambda_1, \lambda_2 \cdots \lambda_n$ are n eigen values.

Now we any polynomial can be written as in form of its root →

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = 0 \quad -(1)$$

And let $X_1, X_2 \cdots X_n$ are eigen vector corresponding to $\lambda_1, \lambda_2 \cdots \lambda_n$ respectively

then $AX_i = \lambda_i X_i$

Multiplying eq (1) with $X_1 X_2 \cdots X_n$

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4. (b) Justify by using Roll's theorem or mean value theorem that there is no number k for which the equation $x^3 - 3x + k = 0$ has two distinct solutions in the interval $[-1,1]$. (15)

Given eqn: $x^3 - 3x + k = 0$.

Let I be interval $[-1, 1]$ or

let $f(x) = x^3 - 3x + k$.

let $f(x) = 0$ has two distinct solution in I.



at x_1, x_2, $x_1 < x_2$

then By Rolle's theorem b/w x_1, x_2

as $f(x_1) = f(x_2)$

$\exists$ $c \in (x_1, x_2)$ s.t $f'(c) = 0$

Now $f'(c) = 3x^2 - 3$

$= 3(x^2 - 1) = 0$

$\Rightarrow x = \pm 1$

$\therefore$ $f'(c)$ is zero only for $-1, +1$

and not for any point in b/w

$\therefore$ no such c exist

$\therefore$ our assumption of two distinct root is wrong

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$\therefore$ for any $n \geq k$, we cannot have two distinct solution in $[-1, 1]$

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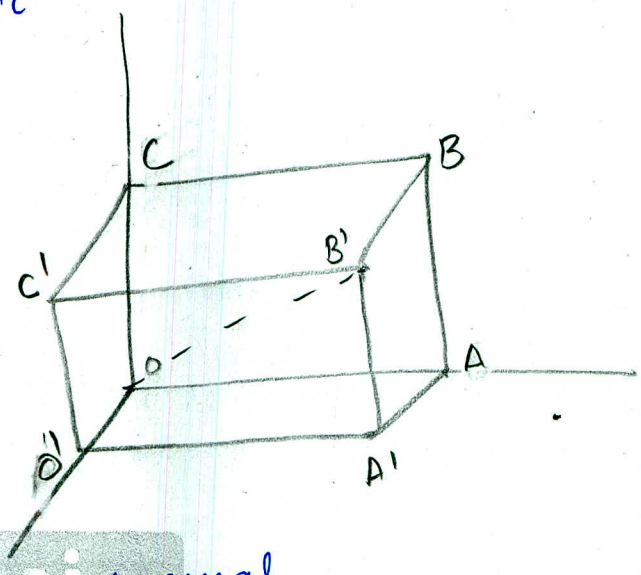
4. (c) Consider a rectangular parallelopiped with edges a, b, c. Obtain the shortest distance between one of its diagonals and an edge which does not intersect this diagonal. (20)

let given OABC O'A'B'C' be parallelopipi with edges a, b c

i.e $OO' = a$

$OA = b$

$OC = c$



let O B' be ~~Any~~ a diagonal

and we will find ~~d~~ its distance from

edge AB

Coordinate of B' is (a, b, c)

Cordnate of A is (0, b, 0)

Coordine of B is (0, b, c)

eqn of line AB is given by →

$$\frac{x}{0} = \frac{y-b}{0} = \frac{z}{c}$$

or $\frac{x}{0} = \frac{y-b}{0} = \frac{z}{1}$

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Now, eqn of line OB' is →

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

Any point on AB is $(0, b, t_1)$

any point on OB' is (at_2, bt_2, ct_2)

let above point as point of shorted distance

then dir will be $= (at_2, bt_2 - b, ct_2 - t_1)$

& $\perp$ to both OB' & AB

i.e. $(at_2)\,a + b(t_2 - 1)b + (ct_2 - t_1)c = 0$

$$t_2(a^2 + b^2 + c^2) - b^2 - ct_1 = 0.$$

$$t_2 = \frac{b^2 + ct_1}{a^2 + b^2 + c^2}.$$

and $\perp$ to OB' :.

$1.(ct_2 - t_1) = 0 \Rightarrow t_1 = ct_2.$

from above two equation

$$t_2(a^2 + b^2 + c^2) - b^2 - c^2 t_2 = 0$$

$$t_2 = \frac{b^2}{a^2 + b^2}, \quad t_1 = \frac{b^2 c}{a^2 + b^2}.$$

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$\therefore$ Shortest distance will be $\rightarrow$

$$d^2 = a^2t_2^2 + (bt_2 - b)^2 + (ct_2 - t_1)^2$$

$$= a^2 \frac{b^4}{(a^2+b^2)^2} + b^2 \left(\frac{a^4}{(a^2+b^2)^2}\right)^2$$

$$= a^2b^2 \frac{(a^2+b^2)}{(a^2+b^2)^2}$$

$$\boxed{d^2 = \frac{a^2b^2}{a^2+b^2}}$$

$\therefore$ Shortest distance $= \boxed{\frac{ab}{\sqrt{a^2+b^2}}}$

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SECTION-B

5. (a) Find the Laplace transform of $\tan^{-1}\frac{1}{s}$. (10)

Let $F(s) = \tan^{-1}(1/s) = \frac{\pi}{2} - \tan^{-1}(s)$

& let $L\{F(s)\} = f(p)$.

We know that

$$L\{F'(s)\} = f(p) - pF(0)$$

Now, $F'(s) = \frac{1}{1+1/s^2} \cdot \frac{-1}{s^2} = \frac{-1}{s^2+1}$

$$= \frac{-1}{2i}\left(\frac{(s+i)-(s-i)}{s^2+1}\right)$$

$$= \frac{-1}{2i}\left(\frac{1}{s+i} - \frac{1}{s+i}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{s+i} - \frac{1}{s-i}\right)$$

Now $f(p) = L\left\{\frac{\pi}{2} - \tan^{-1}s\right\}$

$$= \frac{\pi}{2p} - L\{\tan^{-1}s\}$$

$L\{\tan^{-1}s\} = L\left\{s - \frac{s^3}{3} + \frac{s^5}{5} - \frac{s^7}{7} + \cdots\right\}$

expansion of $\tan^{-1}s$

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$$= \mathcal{L}\left\{\frac{\lfloor 2}{p^2} - \frac{\lfloor 4}{3 \cdot p^4} + \frac{\lfloor 6}{5 \cdot p^6} - \frac{\lfloor 8}{7 \cdot p^8}\right\}$$

$$= \frac{1}{p^2} - \frac{2!}{p^4} + \frac{4!}{p^6} - \frac{6!}{p^8} + \cdots$$

$$= \frac{1}{p^2}\left(1 - \frac{2!}{p^2} + \frac{4!}{p^4} - \frac{6!}{p^6} + \cdots\right)$$

$$\therefore \boxed{f(p) = \mathcal{L}\{\tan^{-1} 1/s\} = \frac{\pi}{2p} - \frac{1}{p^2}\left(1 - \frac{2!}{p^2} + \frac{4!}{p^4} - \frac{6!}{p^6} + \cdots\right)}$$

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5. (b) Determine the constant 'a' so that the vector $V = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal. (10)

Given: $V = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$

V is solenoidal if $\nabla \cdot V = 0$

ie $\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$

$\Rightarrow 1 + 1 + a = 0$

$\Rightarrow \boxed{a = -2}$

$\therefore$ for $a = -2$

V is Solenoida

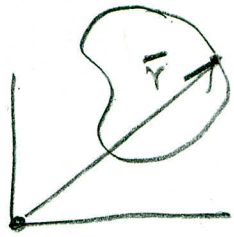
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5. (c) Evaluate $\iint_S r.n\,dS$, where S is a closed surface. (10)

let $I = \iint_S r \cdot n \, dS$

& S be any closed surface



Now let V be volume enclosed by Surface S.

then By Gauss divergence theorem,

$$\iiint_V (\nabla \cdot r)\, dV = \iint_S r \cdot n \, dS.$$

& $\nabla \cdot r = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

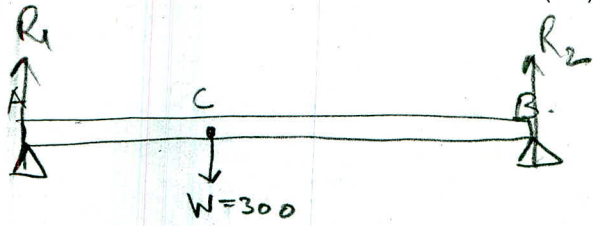
$\therefore \iint_S r \cdot n \, ds = \iiint_V 3 \, dV = 3V$

$\therefore$ $\boxed{\iint_S r \cdot n \, ds = 3V}$, where V is Volume enclosed by Surface S.

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5. (d) A horizontal rod, 1.8 m long, whose weight may be neglected rests on two supports at its extremities; a body of weight 300kg is suspended from the rod at a distance of .75 m from one end; find the reaction at each point of support. If one support could only bear a thrust equal to the weight of 50kg, what is the greatest distance from the other support at which the body could be suspended? (10)



let AB be the horizontal rod resting on support at A & B.

Weight of 300kg is suspended at C where AC = 0.75m & given that AB = 1.8m

let R_1 & R_2 be reaction at A & B.

then Balancing Vertical force →

$$R_1 + R_2 = 300 \quad \text{---(1)}$$

Now taking moment about A. which will be 0 because system is in equilibrium

$$-W(AC) + R_2(AB) = 0$$

$$R_2(1.8) = 300 \times (0.75)$$

$$R_2 = \frac{300 \times 0.75}{1.8} = \frac{25}{3} \times 15 = 125$$

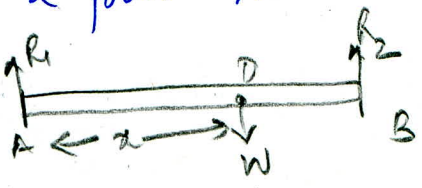
$\therefore$ $\boxed{R_2 = 125}$ & $\boxed{R_1 = 175}$

Reaction at A = 175 kg

B = 125 kg.

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Let support at B can support only 50kg & we place W at distance x from A.



then $R_1 + R_2 = 300$

& Moment about A will be $\circlearrowleft$

$$-W(x) + R_2(1.8) = 0$$

$$\Rightarrow x = \frac{R_2(1.8)}{W} = \frac{50 \times 1.8}{300_6} \quad (\because R_2 \leq 50)$$

$$\Rightarrow x = 0.3.$$

$\therefore$ W should be suspended from **0.3 m** from other end i.e. A.

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5. (e) Show that if the displacements of a particle in a straight line is expressed by the equation $x = a\cos nt + b\sin nt$, it describes a simple harmonic motion whose amplitude is $\sqrt{a^2+b^2}$ and period $2\pi/n$. (10)

Given that, displacement of particle is given by $x = a\cos nt + b\sin nt$

Show → it describe SHM. with Amplitude $= \sqrt{a^2+b^2}$ & time period $= 2\pi/n$

Given: $x = a\cos nt + b\sin nt$

$$\frac{dx}{dt} = n(-a\sin nt + b\cos nt)$$

$$\frac{d^2x}{dt^2} = n^2(-a\cos nt - b\sin nt)$$

$$\boxed{\frac{d^2x}{dt^2} = -n^2x}$$ which is eq^n of SHM

∴ particle describe simple harmonic motion

On comparing with standard SHM equation-

$$\frac{d^2a}{dt^2} = -\mu^2 x \qquad \mu = n$$

& time period $= \frac{2\pi}{\sqrt{\mu}} = \boxed{\frac{2\pi}{n}}$

And, amplitude is displacement when velocity $\frac{dx}{dt} = 0$

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i.e $\frac{dx}{d\theta} = n(-a\sin nt + b\cos nt) = 0$

$\Rightarrow a\sin nt = b\cos nt$

$\Rightarrow \tan(nt) = b/a$

Value of x for this time →

$x = a\cos nt + b\sin nt$

$= a\frac{a}{\sqrt{a^2+b^2}} + \frac{b\cdot b}{\sqrt{a^2+b^2}} = \frac{a^2+b^2}{\sqrt{a^2+b^2}}$

$\therefore$ $\boxed{\text{Amp} = \sqrt{a^2+b^2}}$

NEXT IAS

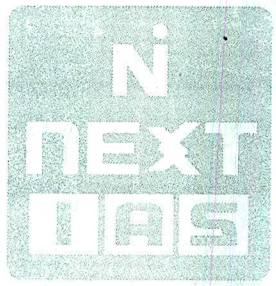
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6. (a) Obtain the Clairaut's form of the differential equation $\left(x\frac{dy}{dx}-y\right)\left(y\frac{dy}{dx}+x\right)=a^2\frac{dy}{dx}$. Also find its general solution. (20)

NEXT IAS

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6. (b) Show that the greatest angle through which a person can oscillate on a swing, the ropes of which can support twice the person's weight at rest is 120°. If the ropes are strong enough and he can swing through 180° and if v is his speed at any point, prove that the tension in the rope at that point is $\frac{3mv^2}{2l}$ where m is the mass of the person and l the length of the rope. (15)

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6. (c) Two equal uniform rod AB and AC, of length a each, are freely joined at A, and are placed symmetrically over two smooth pegs on the same horizontal level at a distance c apart ($3c < 2a$). A weight equal to that of a rod is suspended from the joint A. In the position of equilibrium, find the inclination of either rod with the horizontal by the principle of virtual work. (15)

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7. (a) Find the solution of the differential equation $2x^2y\left(\frac{dy}{dx}\right) = \tan^2\left(x^2y^2\right) - 2xy^2$. What will be the definite value of the arbitrary constant, appearing in the solution, on coordinate axes? (15)

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7. (b) Verify Stokes theorem for $\vec{F} = x\hat{i} + z^2\hat{j} + y^2\hat{k}$ over the plane surface: $x + y + z = 1$, $x + y + z = 1$, lying in the first octant. (20)



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7. (c) If $F = \mu(u^2 - au^3)$, where $a > 0$, be the central force per unit mass and a particle is projected from an apse at a distance a from the centre of force with a velocity $\sqrt{\frac{\mu c}{a^2}}$, where $a > c$, prove that the other apsidal distance of the orbit is $\frac{a(a+c)}{a-c}$.

(15)



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8. (a) Solve: $y'' + 2xy' + (x^2+1)y = x^3 + 3x$. (15)

Given D.E: $y'' + 2xy' + (x^2+1)y = x^3 + 3x$

On comparing with $y' + Py' + Qy = R$

$P = 2x \quad Q = x^2+1 \quad R = x^3+3x$

Consider $u = e^{\int -P/2\, dx}$

$= e^{\int -x\, dx} = e^{-x^2/2}$

we will use Normal form to solve.

& transform equation to $\frac{d^2v}{dx^2} + Iv = S$.

where $I = Q - \frac{P^2}{4} - \frac{P'}{2}$ & $S = R$

and $y = uv$, ~~dv~~

In this Case, $I = x^2+1 - \frac{4x^2}{4} - \frac{2}{2}$.

$= 0$

$\therefore$ d differential eq^n for v will be

$\frac{d^2v}{dx^2} + 0 = x^3 + 3x$

On integrating w.r.t x

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$$\frac{dv}{dx} = \int (x^3 + 3x)\,dx + c_1$$

$$= \frac{x^4}{4} + \frac{3x^2}{2} + c_1$$

$$\therefore v = \int \left(\frac{x^4}{4} + \frac{3x^2}{2} + c_1\right) dx + c_2$$

$$= \frac{x^5}{20} + \frac{x^3}{2} + c_1 x + c_2$$

$\therefore$ general solution of DE is

$$\boxed{y = e^{-x^2/2}\left[\frac{x^5}{20} + \frac{x^3}{2} + c_1 x + c_2\right]}$$

where c_1, c_2 are arbitrary constant

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8. (b) Find the curvature and torsion of the curve $\bar{r} = a(u - \sin u)\hat{i} + a(1 - \cos u)\hat{j} + bu\hat{k}$. (15)

$$\bar{r} = a(u - \sin u)\hat{i} + a(1 - \cos u)\hat{j} + bu\hat{k}$$

We know that, $k(\text{Curvature}) = \dfrac{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|}{\left|\frac{dr}{dt}\right|^3}$.

& $\tau(\text{torsion}) = \dfrac{\left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right]}{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|^2}$

$$\frac{dr}{dt} \Rightarrow \frac{d\bar{r}}{du} = a(1 - \cos u)\hat{i} + a\sin u\hat{j} + b\hat{k}$$

$$\frac{d^2\bar{r}}{du^2} = a\sin u\,\hat{i} + a\cos u\,\hat{j}$$

$$\frac{d^3\bar{r}}{du^3} = a\cos u\,\hat{i} - a\sin u\,\hat{j}$$

Now, $$\frac{d\bar{r}}{du} \times \frac{d^2r}{du^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a(1-\cos u) & a\sin u & b \\ a\sin u & a\cos u & 0 \end{vmatrix}$$

$$= -ab\cos u\,\hat{i} + ab\sin u\,\hat{j} + \hat{k}\left(a^2|\cos u - \cos^2 u - \sin^2 u|\right)$$

$$= -ab\cos u\,\hat{i} + ab\sin u\,\hat{j} + a^2(\cos u - 1)\hat{k}$$

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$$\left|\frac{d\bar{r}}{du} \times \frac{d^2\bar{r}}{du^2}\right| = a^2b^2 + a^4(1-\cos u)^2$$

$$\left|\frac{d\bar{r}}{dt}\right|^2 = a^2(1-\cos u)^2 + a^2\sin^2 u + b^2$$

$$= a^2 + b^2 + a^2 - 2a^2\cos u$$

$$= b^2 + 2a^2(1-\cos u)$$

Now, curvature $k = \dfrac{a^2b^2 + a^4(1-\cos u)^2}{\left[b^2 + 2a^2(1-\cos u)\right]^{3/2}}$.

$$\boxed{k = \frac{a^2\left[b^2 + a^2(1-\cos u)^2\right]}{\left[b^2 + 2a^2(1-\cos u)\right]^{3/2}}}$$

Now $\dfrac{d^3r}{du^3} \cdot \left(\dfrac{d\bar{r}}{du} \times \dfrac{d^2\bar{r}}{du^2}\right) = -ab\cos u\,(a\cos u) + ab\sin u\,(-a\sin u) + 0$

$$= -a^2b\,(1)$$

$\therefore$ $$\tau = \frac{-a^2b}{\left[a^2b^2 + a^4(1-\cos u)^2\right]^2}$$

$$= \frac{-a^2b}{a^4\left(b^2 + a^2(1-\cos u)^2\right)^2} = \frac{-b}{\cdots}$$

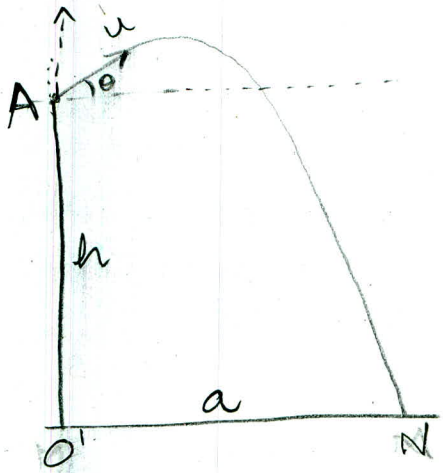
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$$\therefore \boxed{c = \frac{-b}{a^2\left(b^2 + a^2(1-\cos u)^2\right)^2}}$$

NEXT IAS

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8. (c) Prove that the velocity required to project a particle from a height h to fall at a horizontal distance a from a point of projection is at least equal to $\sqrt{g\left[\sqrt{a^2+h^2}-h\right]}$. (20)



let A be point of projection at height h from base O'.

velocity of projection = u

& angle = θ

Now, let particle fall at N with (a) distance from O'

Considering A as origin eqn of projectile is given by →

$$y = x\tan\theta - \frac{g}{2}\frac{x^2}{(u\cos\theta)^2}.$$

∵ particle passes through N. $(a, -h)$

i.e. $-h = a\tan\theta - \frac{g a^2}{2u^2}\sec^2\theta$

$$\frac{g a^2}{2u^2}\sec^2\theta = a\tan\theta + h$$

$$\frac{2u^2}{ga^2} = \frac{\sec^2\theta}{a\tan\theta + h}$$

$$\Rightarrow u^2 = \frac{ga^2}{2}\left(\frac{\sec^2\theta}{a\tan\theta + h}\right)$$

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$$h = -a\tan\theta + \frac{ga^2}{2u^2} + \frac{ga^2}{2u^2}\tan^2\theta$$

$$\tan^2\theta - \frac{2u^2}{ga}\tan\theta + 1 - \frac{2hu^2}{ga^2} = 0$$

for a valid value of θ its discriminant should be +ve

i.e $B^2 - 4AC \geq 0$

$$\Rightarrow \frac{4u^4}{g^2a^2} - 4\left(1 - \frac{2hu^2}{ga^2}\right) \geq 0$$

$$\Rightarrow \frac{4u^4}{g^2a^2} + \frac{8hu^2}{ga^2} - 4 \geq 0 \quad \text{--- (1)}$$

discriminant of quadratic in u^2 is

$$B^2 - 4AC = \frac{64h^2}{g^2a^4} + \frac{16 \cdot 4 \cdot 0}{g^2a^2}$$

$$= \frac{64}{g^2a^4}(h^2 + a^2) \geq 0.$$

Now, Soln of u^2 for equality of eq (1)

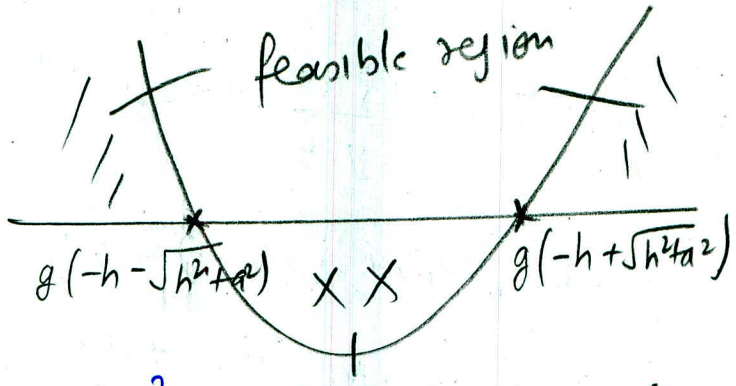
$$u^2 = \frac{-\frac{8h}{ga^2} \pm \sqrt{\frac{64}{g^2a^4}(h^2+a^2)}}{8/g^2a^2}$$

$$= \frac{8}{ga^2}\frac{\left(-h \pm \sqrt{h^2+a^2}\right)}{8/g^2a^2}$$

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$$u^2 = g\left(-h \pm \sqrt{h^2+a^2}\right)$$

It represent



Non feasible as per eqn (1)

$\therefore$ least value of u^2

to satisfy eq (1) and $u^2 > 0$

so $\boxed{u^2 = g\left(\sqrt{h^2+a^2} - h\right)}$

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SPACE FOR ROUGH WORK

$\tan^{-1} 1/s$.

$e^{-at} \tan^{-1} 1/s$.

$L\{F'(t)\} =$

$L^{-1}\{F'(p)\} = \sin t$.

$L^{-1}(F(t$

$\tan^{-1} s$

$= s - \frac{s^3}{3} + \frac{s^5}{5} + \frac{s^7}{7} \ldots$

$\frac{[2)}{p^2} - \frac{3!}{p^4}$

NEXT IAS